

ON SOBOLEV INSTABILITY OF THE INTERIOR PROBLEM OF TOMOGRAPHY

M. Bertola^{†1} A. Katsevich^{*2} and A. Tovbis^{*3}

[†] *Centre de recherches mathématiques, Université de Montréal
C. P. 6128, succ. centre ville, Montréal, Québec, Canada H3C 3J7 and
Department of Mathematics and Statistics, Concordia University
1455 de Maisonneuve W., Montréal, Québec, Canada H3G 1M8*

^{*} *University of Central Florida Department of Mathematics
4000 Central Florida Blvd. P.O. Box 161364 Orlando, FL 32816-1364*

E-mail: bertola@mathstat.concordia.ca, Alexander.Katsevich@ucf.edu, Alexander.Tovbis@ucf.edu

Abstract

In this paper we continue investigation of the interior problem of tomography that was started in [BKT13]. As is known, solving the interior problem with prior data specified on a finite collection of intervals I_i is equivalent to analytic continuation of a function from I_i to an open set \mathbf{J} . In the paper we prove that this analytic continuation can be obtained with the help of a simple explicit formula, which involves summation of a series. Our second result is that the operator of analytic continuation is not stable for any pair of Sobolev spaces regardless of how close the set \mathbf{J} is to I_i . Our main tool is the singular value decomposition of the operator \mathcal{H}_e^{-1} that arises when the interior problem is reduced to a problem of inverting the Hilbert transform from incomplete data. The asymptotics of the singular values and singular functions of \mathcal{H}_e^{-1} , the latter being valid uniformly on compact subsets of the interior of I_i , was obtained in [BKT13]. Using these asymptotics we can accurately measure the degree of ill-posedness of the analytic continuation as a function of the target interval \mathbf{J} . Our last result is the convergence of the asymptotic approximation of the singular functions in the $L^2(I_i)$ sense.

1 Introduction

Suppose one is interested in imaging a small region of interest (ROI) inside an object using tomography. In order to acquire a complete data set that enables stable reconstruction, one needs to send multiple x-rays through the object from many different directions. In particular, the x-rays that do not pass through the ROI are required as well. The interior problem of tomography arises when only the x-rays through the ROI are measured. In this case the tomographic data are incomplete, and image reconstruction becomes a challenging problem. In what follows, image reconstruction from x-ray data tailored to an ROI will be called the interior problem, and the corresponding data will be called interior data. Practical importance of the interior problem is clear, since tailoring the x-ray exposure to an ROI results in a reduced x-ray dose to the patient in medical applications of tomography. See [WY13] for a nice review of the state of the art in interior tomography.

One of the most powerful tools for investigating the interior problem from the theoretical point of view is the Gelfand-Graev formula, which relates the tomographic data of an object with its

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one-dimensional Hilbert transform along lines [GG91]. With the help of this formula, the interior problem of tomography can be reduced to the problem of inverting the Hilbert transform from incomplete data.

Pick any line L through the object. We regard L as the x -axis. Fix some $2g+2$, $g \in \mathbb{N}$, distinct points a_i on L : $a_i < a_{i+1}$, $i = 1, 2, \dots, 2g+1$. Points a_1 and a_{2g+2} mark the boundaries of the support of f along L . Points a_2 and a_{2g+1} mark the boundaries of the ROI along L . Consider the Finite Hilbert Transform (FHT)

$$(\mathcal{H}f)(x) := \frac{1}{\pi} \int_{a_1}^{a_{2g+2}} \frac{f|_L(y)}{y-x} dy, \quad f|_L \in L^2([a_1, a_{2g+2}]). \quad (1.1)$$

Here $f|_L$ is the restriction of f to L , and $\mathcal{H}f$ is the one-dimensional Hilbert transform of $f|_L$. Throughout the paper the line L is always the same, so with some abuse of notation we write f instead of $f|_L$. In the case of interior tomographic data, the Gelfand-Graev formula allows computation of $\mathcal{H}f$ only on $[a_2, a_{2g+1}]$, but not on all $[a_1, a_{2g+2}]$. Thus the interior problem of tomography is reduced to finding f inside the ROI, i.e. on $[a_2, a_{2g+1}]$, by solving the equation

$$(\mathcal{H}f)(x) = \varphi(x), \quad x \in [a_2, a_{2g+1}]. \quad (1.2)$$

Consider the operator $\mathcal{H} : L_2([a_1, a_{2g+2}]) \rightarrow L_2([a_2, a_{2g+1}])$. Unique recovery of f on $[a_2, a_{2g+1}]$ is impossible since \mathcal{H} has a non-trivial kernel (see [KT12] for its complete description). Therefore, to achieve unique recovery the data φ should be augmented by some additional information. One type of information that guarantees uniqueness is the knowledge of f on some interval or intervals inside $[a_2, a_{2g+1}]$. This is the so-called interior problem with prior knowledge ([YYW07, KCND08, CNDK08, WY13]) that will be considered below. Let us assume that f is known on the intervals

$$I_i := [a_3, a_4] \cup [a_5, a_6] \cup \dots \cup [a_{2g-1}, a_{2g}], \quad (1.3)$$

which we call “interior” (inside the ROI). Denote by $I_e := [a_1, a_2] \cup [a_{2g+1}, a_{2g+2}]$ the remaining “exterior” intervals (they are outside the ROI). Applying the FHT inversion formula (see e.g. [OE91]), we get

$$f(y) = -\frac{w(y)}{\pi} \left(\int_{a_1}^{a_2} + \int_{a_{2g+1}}^{a_{2g+2}} \right) \frac{\varphi(x)}{w(x)(x-y)} dx - \frac{w(y)}{\pi} \int_{a_2}^{a_{2g+1}} \frac{\varphi(x)}{w(x)(x-y)} dx, \quad (1.4)$$

where $w(x) := \sqrt{(a_{2g+2}-x)(x-a_1)}$ and $\varphi(x) = (\mathcal{H}f)(x)$, $x \in [a_1, a_{2g+2}]$.

The left side of (1.4) is known on I_i . The last integral on the right is known everywhere. Combining these known quantities we get an integral equation:

$$(\mathcal{H}_e^{-1}\varphi)(y) := -\frac{w(y)}{\pi} \int_{I_e} \frac{\varphi(x)}{w(x)(x-y)} dx = \psi(y), \quad y \in I_i, \quad (1.5)$$

where

$$\psi(y) = f(y) + \frac{w(y)}{\pi} \int_{a_2}^{a_{2g+1}} \frac{\varphi(x)}{w(x)(x-y)} dx, \quad y \in I_i \quad (1.6)$$

is a known function.

The main problem we study in this paper is the stability of finding f from the data. Several approaches to finding f on $[a_2, a_{2g+1}]$ are possible. The first one consists of two steps. In step 1 we solve equation (1.5) for $\varphi(x)$ on I_e . In step 2 we substitute the computed $\varphi(x)$ into (1.4) and recover $f(y)$ on $[a_2, a_{2g+1}]$. It is clear that solving (1.5), i.e. inverting \mathcal{H}_e^{-1} , is the most unstable step. Consider the operator \mathcal{H}_e^{-1} in (1.5) as a map between two weighted L^2 -spaces:

$$\mathcal{H}_e^{-1} : L^2(I_e, 1/w) \rightarrow L^2(I_i, 1/w). \quad (1.7)$$

Its adjoint is the Hilbert transform:

$$(\mathcal{H}_i \psi)(x) := \frac{1}{\pi} \int_{I_i} \frac{\psi(y)}{y-x} dy, \quad x \in I_e. \quad (1.8)$$

In [BKT13] the authors studied the singular value decomposition (SVD) for the operator \mathcal{H}_e^{-1} . Namely, we were interested in the singular values $2\lambda = 2\lambda_n > 0$, $n \in \mathbb{N}$, and the corresponding left and right singular functions $f = f_n$, $h = h_n$, satisfying

$$\begin{aligned} (\mathcal{H}_e^{-1})h(y) &= -\frac{w(y)}{\pi} \int_{I_e} \frac{h(x)}{w(x)(x-y)} dx = 2\lambda f(y), \quad y \in I_i, \\ (\mathcal{H}_i f)(x) &= \frac{1}{\pi} \int_{I_i} \frac{f(y)}{y-x} dy = 2\lambda h(x), \quad x \in I_e. \end{aligned} \quad (1.9)$$

See (2.1)–(2.3) and Theorem 2.1, which show that the SVD is well-defined. It is well known that the rate at which λ_n 's approach zero is related with the ill-posedness of inverting \mathcal{H}_e^{-1} . Because of the symmetry $(\lambda, f, h) \Leftrightarrow (-\lambda, -f, h)$ of (1.9), we are interested only in positive λ_n . The main result of the paper [BKT13] is the large n asymptotics of λ_n , f_n and h_n .

Let us introduce a $g \times g$ matrix \mathbb{A} by

$$(\mathbb{A})_{kj} = 2 \int_{a_{2k}}^{a_{2k+1}} \frac{z^{j-1} dz}{R(z)}, \quad k = 1, \dots, g-1, \quad \text{and} \quad (\mathbb{A})_{gj} = 2 \int_{a_1}^{a_{2g+2}} \frac{z^{j-1} dz}{R_+(z)}, \quad j = 1, \dots, g, \quad (1.10)$$

where $R(z) = \prod_{j=1}^{2g+2} (z - a_j)^{\frac{1}{2}}$ is an analytic function on $\mathbb{C} \setminus (I_e \cup I_i)$ behaving as z^{g+1} at infinity, and define

$$\tau_{11} = -2 \sum_{j=1}^g (\mathbb{A}^{-1})_{j1} \int_{I_e} \frac{z^{j-1} dz}{R_+(z)}. \quad (1.11)$$

Here and throughout the paper the subscripts \pm routinely denote limiting values of functions (vectors, matrices) from the left/right side of corresponding oriented arcs. In particular, R_+ means the limiting value of R on $I = I_e \cup I_i$ from $\Im z > 0$. We also want to note that, according to the well-known Riemann's Theorem on periods of holomorphic differentials ([FK92], τ_{11} is a purely imaginary number with positive imaginary part. Then the asymptotics of λ_n is given by ([BKT13])

$$\lambda_n = e^{-\frac{i\pi}{\tau_{11}} n + \mathcal{O}(1)}, \quad n \rightarrow \infty. \quad (1.12)$$

The asymptotics of the singular functions from [BKT13] is described in Section 2 of this paper. An alternative approach to the analysis of SVD for the Hilbert transform with incomplete data is developed in [Kat10, Kat11, KT12, AAK14].

The very rapid decay of singular values in (1.12) indicates that finding φ from ψ is very unstable. This, however, does not imply that finding f on $[a_2, a_{2g+1}]$ is unstable, since f is computed by applying a smoothing operator to φ . The second approach to finding f is based on the observation that the function ψ defined by (1.6) is analytic in $\mathbb{C} \setminus I_e$ (cf. (1.5)). Hence, analytically continuing ψ from I_i to (a_2, a_{2g+1}) , we can find f using (1.6) with $y \in (a_2, a_{2g+1})$. Note that any method that gives f on (a_2, a_{2g+1}) is equivalent to analytic continuation of ψ in view of (1.6). *Thus, analytic continuation of ψ is at the heart of any method for solving the interior problem of tomography with prior knowledge.*

In this paper we obtain two results regarding the analytic continuation of ψ . We show that this analytic continuation can be obtained with the help of a simple explicit formula, which involves summation of a series, see Corollary 3.4. We prove that the series is absolutely convergent if ψ is in the range of \mathcal{H}_e^{-1} . We also analyze stability of this analytic continuation. Intuitively, it is clear that the farther away from I_i we continue ψ the less stable the procedure becomes. Our second result is that the operator of analytic continuation is not stable for any pair of Sobolev spaces: $H^{s_1}(I_i) \rightarrow H^{-s_2}(J)$, where J is any open set containing I_i . In other words, the procedure is unstable no matter how close to I_i we perform the continuation. This is an interesting result, because earlier related results indicated that finding f might be stable [DNCK06, KCND08].

The paper is organized as follows. Since the derivation of our main results strongly depends on the results in [BKT13], the latter are briefly reviewed in Section 2. The analytic continuation of ψ and its instability in the Sobolev spaces are established in Section 3. Loosely speaking, this result shows that no matter how many derivatives are required of ψ , the continuation is not stable. The availability of asymptotics of singular values and singular functions allows us to accurately estimate the degree of instability of the continuation. In Section 3 we introduce a Hilbert space \mathcal{A} of functions defined on I_i with the help of an exponentially growing weight. We show how fast this weight must grow in order to ensure that the analytic continuation from I_i to an open set \mathbf{J} be a continuous map from $\mathcal{A} \rightarrow L^2(\mathbf{J})$. Thus, this rate of growth measures the degree of ill-posedness of the analytic continuation as a function of the target interval \mathbf{J} .

In [BKT13] it is shown that the asymptotic approximations to the exact singular functions f_n are valid uniformly on compact subsets of the interior of I_i as $n \rightarrow \infty$. In Section 4 we show that these approximations are also valid in the $L^2(I_i)$ sense as well. This is the third result obtained in this paper. We do not consider the other set of singular functions that are defined on I_e , since they are not needed for the analytic continuation of ψ . The main idea of the approach in [BKT13] is to reduce the SVD problem (1.9) to a matrix Riemann-Hilbert problem (RHP), which, in turn, is asymptotically reduced to a simpler RHP. That simpler (model) RHP has an explicit solution, which can be expressed in terms of the Riemann Theta function. A brief review of the reduction to the model RHP and certain related results from [BKT13] are contained in Appendix A. Some technical lemmas related to the approximation of singular functions on $[a_1, a_{2g+2}] \setminus I$ and on I_i that are needed in Sections 3 and 4 are proven in Appendix B.

2 Brief review of main results of [BKT13]

This section contains a brief review of major results of [BKT13]. For convenience, most of the statements below are provided with direct references (in square brackets) to the corresponding results of [BKT13].

The SVD system (1.9) can be represented as

$$\begin{aligned}(H_e^{-1}\hat{h})(y) &:= \frac{\sqrt{w(y)}}{2\pi i} \int_{I_e} \frac{\hat{h}(x)}{\sqrt{w(x)}(x-y)} dx = \lambda \hat{f}(y), \quad y \in I_i, \\ (H_i \hat{f})(x) &:= \frac{1}{2\pi i} \frac{1}{\sqrt{w(x)}} \int_{I_i} \frac{\hat{f}(y) \sqrt{w(y)}}{(y-x)} dy = \lambda \hat{h}(x), \quad x \in I_e,\end{aligned}\tag{2.1}$$

where $\hat{h} = \frac{h}{\sqrt{w}} \in L^2(I_e)$, $\hat{f} = \frac{if}{\sqrt{w}} \in L^2(I_i)$, and the operators H_e^{-1} , H_i act on the corresponding unweighted L^2 spaces. It can be checked directly that the triple $(\lambda, \hat{f}, \hat{h})$ satisfies the system (2.1) if and only if λ, ψ is the eigenvalue/eigenvector of the integral operator $(\hat{K}\phi)(z) = \int_I K(z, x)\phi(x)dx$ from $L^2(I)$ to $L^2(I)$, where

$$K(z, x) = \frac{w^{\frac{1}{2}}(x)w^{-\frac{1}{2}}(z)\chi_e(z)\chi_i(x) + w^{\frac{1}{2}}(z)w^{-\frac{1}{2}}(x)\chi_i(z)\chi_e(x)}{2i\pi(x-z)}, \quad \psi = \hat{f}(z)\chi_i(z) + \hat{h}(z)\chi_e(z).\tag{2.2}$$

(Here and henceforth $\chi_i(z), \chi_e(z)$ denote the characteristic (indicator) functions of the sets I_i, I_e , respectively.) Thus, the SVD problem for the system (2.1) is reduced to the spectral problem for the integral operator $\hat{K} : L^2(I) \rightarrow L^2(I)$. It follows directly from (2.2) that

$$\hat{K}|_{L^2(I_i)} = H_i, \quad \hat{K}|_{L^2(I_e)} = H_e^{-1}.\tag{2.3}$$

Theorem 2.1. *[Thm.3.1 and Cor.3.8] \hat{K} is a self-adjoint and a Hilbert–Schmidt operator. Moreover, all the eigenvalues of \hat{K} are simple.*

According to Theorem 2.1, the eigenvalues of \hat{K} are real with the only possible point of accumulation $\lambda = 0$. Since the singular values of (2.1) are positive (note the symmetry $(\lambda, \hat{f}, \hat{h}) \mapsto (-\lambda, -\hat{f}, \hat{h})$ in (2.1)), we are interested only in the positive eigenvalues λ_n , $n \in \mathbb{N}$, of \hat{K} , where we order $\lambda_0 > \lambda_1 > \dots > 0$.

Let \hat{L} denote the restrictions of \hat{K} to the interval I_i . Then, according to (2.2), $\hat{L} = H_e^{-1}H_i : L^2(I_i) \rightarrow L^2(I_i)$ is an integral operator with eigenvalues λ_n^2 and eigenfunctions \hat{f}_n , $n \in \mathbb{N}$. It is interesting to note (Lemma 3.6 in [BKT13]) that \hat{L} is a strictly totally positive operator. Then the simplicity of the eigenvalues λ_n^2 of \hat{L} and, thus, of λ_n of \hat{K} in Theorem 2.1, follows from properties of strictly totally positive integral operators (see [Pin96]). Another consequence of this property of \hat{L} is that the singular function \hat{f}_n has exactly n sign changes on I_i , $n = 0, 1, 2, \dots$.

An important object of the spectral theory is the resolvent operator \hat{R} of \hat{K} , defined by

$$(\text{Id} + \hat{R})(\text{Id} - \frac{1}{\lambda}\hat{K}) = \text{Id}.\tag{2.4}$$

The resolvent operator \hat{R} is an integral operator with the kernel of the form

$$R(z, x; \lambda) = \frac{\vec{g}^t(x)\Gamma^{-1}(x; \lambda)\Gamma(z; \lambda)\vec{f}(z)}{2i\pi\lambda(z-x)}, \quad \text{where } \vec{f}(z) := \begin{bmatrix} \frac{i\chi_e(z)}{\sqrt{w(z)}} \\ \frac{\chi_e(z)}{\sqrt{w(z)}\chi_i(z)} \end{bmatrix}, \quad \vec{g}(x) := \begin{bmatrix} -i\sqrt{w(x)}\chi_i(x) \\ \frac{\chi_e(x)}{\sqrt{w(x)}} \end{bmatrix},\tag{2.5}$$

where \vec{g}^t denotes the transposition of \vec{g} and the matrix $\Gamma(z; \lambda)$ satisfies the following Riemann–Hilbert Problem (RHP) 2.2.

Riemann-Hilbert Problem 2.2. Find a 2×2 matrix-function $\Gamma = \Gamma(z; \lambda)$, $\lambda \in \mathbb{C} \setminus \{0\}$, which is analytic in $\mathbb{C} \setminus I$, where $I = I_i \cup I_e$, admits non-tangential boundary values from the upper/lower half-planes that belong to L^2_{loc} in the interior points of I , and satisfies

$$\Gamma_+(z; \lambda) = \Gamma_-(z; \lambda) \begin{bmatrix} 1 & 0 \\ \frac{iw}{\lambda} & 1 \end{bmatrix}, \quad z \in I_i; \quad \Gamma_+(z; \lambda) = \Gamma_-(z; \lambda) \begin{bmatrix} 1 & -\frac{i}{\lambda w} \\ 0 & 1 \end{bmatrix}, \quad z \in I_e, \quad (2.6)$$

$$\Gamma(z; \lambda) = \mathbf{1} + O(z^{-1}) \quad \text{as } z \rightarrow \infty, \quad (2.7)$$

$$\Gamma(z; \lambda) = [\mathcal{O}(1), \mathcal{O}((z - a_j)^{-\frac{1}{2}})], \quad z \rightarrow a_j, \quad j = 1, 2g + 2, \quad (2.8)$$

$$\Gamma(z; \lambda) = [\mathcal{O}(1), \mathcal{O}(\ln(z - a_j))], \quad z \rightarrow a_j, \quad j = 2, 2g + 1, \quad (2.9)$$

$$\Gamma(z; \lambda) = [\mathcal{O}(\ln(z - a_j)), \mathcal{O}(1)], \quad z \rightarrow a_j, \quad j = 3, \dots, 2g. \quad (2.10)$$

Here the endpoint behavior of Γ is described column-wise. We will frequently omit the dependence on λ from notation and write simply $\Gamma(z)$ for convenience.

The latest fact links the resolvent operator \hat{R} for \hat{K} with the RHP for the matrix Γ from (2.5).

Theorem 2.3. [Thm.3.17 and Prop.3.12] The RHP 2.2 has a solution $\Gamma(z; \lambda)$, where $\lambda \in \mathbb{C} \setminus \{0\}$, if and only if λ is not an eigenvalue of \hat{K} . Moreover, for any fixed $\lambda \in \mathbb{C} \setminus \{0\}$ the RHP 2.2 has at most one solution.

Connection of our spectral problem with the RHP 2.2 is remarkable, as the RHP 2.2 is a much more convenient object for rigorous asymptotic analysis (in small λ) than the spectral problem for \hat{K} . The eigenfunctions of \hat{K} corresponding to a fixed eigenvalue λ_n are given by two proportional expressions

$$\phi_{n,j}(z) = \frac{\chi_e(z)}{\sqrt{w(z)}} \operatorname{res}_{\lambda=\lambda_n} \Gamma_{j1}(z; \lambda) \frac{1}{\lambda} + i\sqrt{w(z)}\chi_i(z) \operatorname{res}_{\lambda=\lambda_n} \Gamma_{j2}(z; \lambda) \frac{1}{\lambda}, \quad j = 1, 2, \quad (2.11)$$

in terms of the entries of the matrix $\Gamma(z, \lambda)$, where for every $n \in \mathbb{N}$ at least one of $\phi_{n,j}$ is not identical zero on I .

Once the connection between the spectral problem for \hat{K} and the RHP 2.2 is established, we use the nonlinear steepest descent method of Deift and Zhou to construct an explicit leading order approximation of $\Gamma(z, \lambda)$ as $\lambda \rightarrow 0^+$ in terms of the Riemann Theta functions. Of course, this approximation will not be valid at the eigenvalues λ_n of \hat{K} , as, according to Theorem 2.3, $\Gamma(z, \lambda_n)$ does not exist. However, using the explicit form of the approximate solution, we can find the values $\tilde{\lambda}_n$ for which this approximate solution has singularities. The obtained values $\tilde{\lambda}_n$ will be referred to as “approximate eigenvalues”. It turns out that, indeed, $\tilde{\lambda}_n$ approximate the corresponding λ_n with the accuracy

$$|\varkappa_n - \tilde{\varkappa}_n| = O(\tilde{\varkappa}_n^{-\frac{1}{2}}), \quad (2.12)$$

where $\varkappa_n = -\ln \lambda_n$ and $\tilde{\varkappa}_n = -\ln \tilde{\lambda}_n$ (it will be shown that $\tilde{\varkappa}_n = O(n)$ as $n \rightarrow \infty$).

Let us now consider the asymptotics of singular functions. According to (2.2), the approximation of normalized singular functions can be obtained by replacing rows of the matrix $\Gamma_{jk}(z; \lambda)$, $j, k \in \{1, 2\}$, in (2.11) by the corresponding rows of the approximate solution to the RHP 2.2. To present the approximation formula for singular functions, we need to introduce some notations and a few notions from the theory of compact Riemann surfaces. They will also be helpful for a geometrical interpretation of $\tilde{\varkappa}_n$.

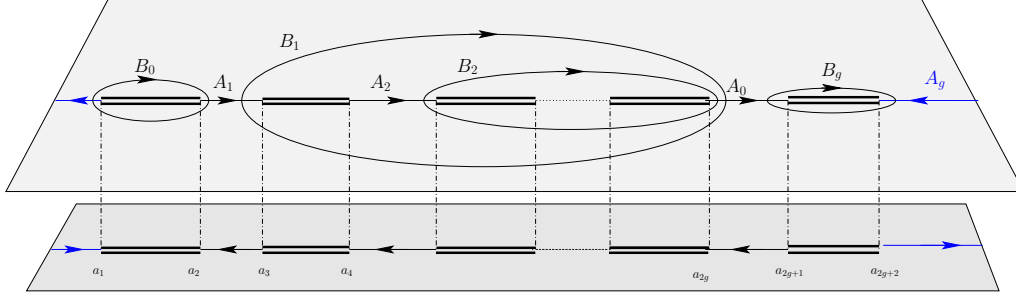


Figure 1: Riemann surface \mathcal{R} with the choice of A and B cycles.

The **Riemann Theta function** associated with a symmetric matrix τ with strictly positive imaginary part (that guarantees convergence) is the function of the vector argument $\vec{z} \in \mathbb{C}^g$ given by

$$\Theta(\vec{z}, \tau) := \sum_{\vec{n} \in \mathbb{Z}^g} \exp \left(i\pi \vec{n}^t \cdot \tau \cdot \vec{n} + 2i\pi \vec{n}^t \vec{z} \right). \quad (2.13)$$

Often the dependence on τ is omitted from the notation. We will consider the matrix τ given by

$$\tau = [\tau_{ij}] = \left[\oint_{B_i} \omega_j d\zeta \right]_{i,j=1,g}, \quad (2.14)$$

where

$$\vec{\omega}^t(z) = [\omega_1(z), \dots, \omega_g(z)] = \frac{[1, \dots, z^{g-1}]}{R(z)} \mathbb{A}^{-1}, \quad (2.15)$$

matrix \mathbb{A} is defined by (1.10), and the loops (cycles) B_i , $i = 1, \dots, g$ are shown in Figure 1.

Theorem 2.4 (Riemann [FK92]). *The matrix τ is symmetric and its imaginary part is strictly positive definite.*

Matrix τ is an important object in the theory of compact Riemann surfaces. Indeed, consider the hyperelliptic Riemann surface \mathcal{R} , defined by the segments $[a_{2k-1}, a_{2k}]$, $k = 1, 2, \dots, g+1$, that form I , with canonical A and B cycles shown in Figure 1. Then $\vec{\omega}(z)dz$ is known as the vector of normalized holomorphic differentials on \mathcal{R} and τ is called the normalized matrix of B -periods of \mathcal{R} . Note that $[\mathbb{A}]_{ji} = \oint_{A_j} \frac{\zeta^{i-1} d\zeta}{R(\zeta)}$, and τ_{11} in (1.11) is the $(1,1)$ entry of the matrix τ .

Remark 2.5. *It follows from (2.14), (2.15) and (1.10) that the entries of the matrix τ are purely imaginary.*

Proposition 2.6. *For any $\lambda, \mu \in \mathbb{Z}^g$, the Theta function has the following properties:*

$$\Theta(\vec{z}, \tau) = \Theta(-\vec{z}, \tau); \quad (2.16)$$

$$\Theta(\vec{z} + \mu + \tau\lambda, \tau) = \exp \left(-2i\pi \lambda^t \vec{z} - i\pi \lambda^t \tau \lambda \right) \Theta(\vec{z}, \tau). \quad (2.17)$$

According to (2.13) and Proposition 2.6, the Theta function is an even function of g complex variables, periodic on the lattice \mathbb{Z}^g and quasi-periodic on the lattice $\tau\mathbb{Z}^g$. A hypersurface $(\Theta) \subset \mathbb{C}^g$, defined by $\Theta(\vec{z}, \tau) = 0$, is called a theta divisor. This is a hypersurface of complex codimension one or real codimension two. According to Proposition 2.6, the theta divisor (Θ) is periodic in \mathbb{Z}^g and $\tau\mathbb{Z}^g$.

Let

$$W = W(\varkappa) = \frac{\varkappa}{i\pi}\tau_1 + 2u(\infty) + \frac{\mathbf{e}_1}{2}, \quad W_0 = \frac{\tau_1}{2} - \frac{\mathbf{e}_1 + \mathbf{e}_g}{2}, \quad (2.18)$$

where τ_1 is the first column of matrix τ ,

$$u(z) = \int_{a_1}^z \bar{\omega}(\zeta) d\zeta, \quad z \in \mathbb{C} \setminus [a_1, \infty), \quad (2.19)$$

is known as the Abel map on \mathcal{R} , and \mathbf{e}_k denotes the k th vector of the standard basis in \mathbb{C}^g . Then $\tilde{\varkappa}_n = -\ln \tilde{\lambda}_n$ are defined by the condition

$$\Theta(W(\varkappa) - W_0) = 0. \quad (2.20)$$

Geometrically, this condition determines the points of intersection of the line $W(\varkappa) - W_0 \subset \mathbb{C}^g$ with the theta divisor. Let us consider this question in a little more details. Direct calculations show that all the terms of $W(\varkappa)$ in (2.18) are real, provided that $\varkappa \in \mathbb{R}$. Thus, the line $\{W(\varkappa) : \varkappa \in \mathbb{R}\} \subset \mathbb{R}^g \subset \mathbb{R}^{2g}$, if we identify \mathbb{C}^g with \mathbb{R}^{2g} . So, the line $W(\varkappa) - W_0$, $\varkappa \in \mathbb{R}$, is a subset of the shifted hyperplane $\Pi = W_0 + \mathbb{R}^g$. Let $(\Theta)_R := (\Theta) \cap \Pi$.

Lemma 2.7. *[Lem.7.5] Each connected component of $(\Theta)_R$ is a smooth $g - 1$ (real) dimensional hypersurface in Π .*

Moreover, since $(\Theta)_R$ is \mathbb{Z}^g periodic on Π , it is sufficient to study $(\Theta)_R$ in a g (real) dimensional torus \mathbb{T}_g . Numerically simulated surfaces $(\Theta)_R \cap \mathbb{T}_g$ for $g = 2, 3$, and their intersections with the line $W(\varkappa) - W_0$ are shown on Figure 2. In the case $g = 2$ we proved that the line $W(\varkappa) - W_0$ has one and only one intersection with $(\Theta)_R$ in \mathbb{T}_2 . It is likely (but not proven yet) that this statement holds for a general $g \in \mathbb{N}$. However, the following lemma is sufficient to obtain the asymptotics (1.12) for λ_n with any $g \in \{2, 3, \dots\}$.

Lemma 2.8. *[Prop.7.11] Let $\varkappa_0 \in \mathbb{R}^+$ and $g \in \{2, 3, \dots\}$. For any $N \in \mathbb{N}$ the number $m(N)$ of intersections of the segment of the line $W(\varkappa) - W_0$, where $\varkappa \in \left[\varkappa_0, \varkappa_0 + \frac{N(g-1)i\pi}{\tau_{11}}\right)$, with $(\Theta)_\mathbb{R}$ is bounded by*

$$(N - 1)(g - 1) \leq m(N) \leq (N + 1)(g - 1). \quad (2.21)$$

Let us now denote

$$\begin{aligned} \mathbf{f}_n &:= W(\tilde{\varkappa}_n) - W_0, \quad \mathbf{g}(z) = \frac{1}{2} - 2 \int_{a_1}^z \omega_1 dz \quad \text{and} \\ d(z) &= \frac{R(z)}{2\pi i} \left(- \sum_{j=1}^{g+1} \int_{a_{2j-1}}^{a_{2j}} \frac{\ln w(\zeta) d\zeta}{(\zeta - z)R_+(\zeta)} + \sum_{j=1}^g \int_{a_{2j}}^{a_{2j+1}} \frac{i\delta_{\mu(j)} d\zeta}{(\zeta - z)R_+(\zeta)} \right), \end{aligned} \quad (2.22)$$

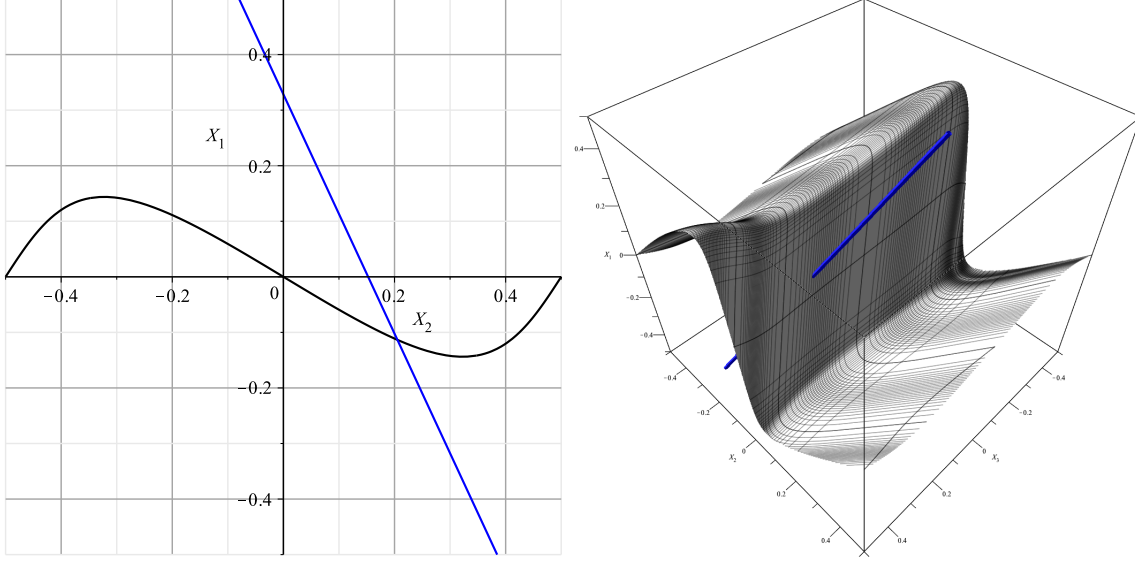


Figure 2: Intersection of the line $W(\kappa) - W_0$ (blue or lighter line) with the theta divisor $(\Theta)_R$ in \mathbb{T}_g , where $g = 2$ (left panel) and $g = 3$ (right panel). On the left panel ($g = 2$) $(\Theta)_R$ is represented by a curve, on the right panel ($g = 3$) $(\Theta)_R$ is represented by a surface. In both cases the point of intersection of $W(\kappa) - W_0$ with $(\Theta)_R$ determines some $\kappa = \tilde{\kappa}_n$.

where: $\mu(g) = 0$ and $\mu(j) = j$ for all $j \neq g$; the vector $\vec{\delta} = [\delta_1, \dots, \delta_{g-1}, \delta_0]^t$ is given by $\vec{\delta} = 2\pi L^{-1} (2u(\infty) - u(a_{2g+2}))$ and

$$L = \begin{bmatrix} 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & 0 & \dots & 0 & -1 \\ & & \ddots & & & \\ & & & \dots & 1 & -1 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}. \quad (2.23)$$

Proposition 2.9. [Prop.4.2] **(1)** $\mathfrak{g}(z)$ satisfies the jump conditions

$$\mathfrak{g}_+(z) + \mathfrak{g}_-(z) = -1 \quad \text{on } I_i, \quad \mathfrak{g}_+(z) + \mathfrak{g}_-(z) = 1 \quad \text{on } I_e, \quad (2.24)$$

$$\text{and} \quad \mathfrak{g}_+(z) - \mathfrak{g}_-(z) = i\Omega_{\mu(j)} \quad \text{on } [a_{2j}, a_{2j+1}], \quad j = 1, \dots, g, \quad (2.25)$$

where $\Omega_0 = \frac{4}{i} \sum_{k=1}^g \int_{a_{2k-1}}^{a_{2k}} \omega_1 dz \in \mathbb{R}$ and $\Omega_j = \frac{4}{i} \sum_{k=1}^j \int_{a_{2k-1}}^{a_{2k}} \omega_1 dz \in \mathbb{R}$. **(2)** The function $d(z)$ given by (2.22) is analytic on $\mathbb{C} \setminus [a_1, a_{2g+2}]$ (in particular, analytic at infinity) and satisfies the jump conditions

$$d_+ + d_- = -\ln w \quad \text{on } I, \quad d_+ - d_- = i\delta_{\mu(j)} \quad \text{on } c_j, \quad [a_{2j}, a_{2j+1}], \quad j = 1, \dots, g. \quad (2.26)$$

Let

$$r(z) := \sqrt[4]{\frac{\prod_{j \in J} (z - a_j)}{\prod_{\ell \in J'} (z - a_\ell)}}, \quad z \in \mathbb{C} \setminus [a_1, a_{2g+2}], \quad (2.27)$$

where $J = \{1, 5, 7, 9, 11, \dots, 2g-1\}$ and $J' = \{1, 2, 3, \dots, 2g+2\} \setminus J$ (so that $|J| = g-1$ and $|J'| = g+3$). The function $r(z)$ is defined so that it is analytic in $\mathbb{C} \setminus [a_1, a_{2g+2}]$ and at infinity behaves like $\frac{1}{z}$.

Let

$$\begin{aligned} \Upsilon^{(j)}(z; \mathbf{f}_n) &= (-1)^j \sqrt{\frac{\Theta(W_0 + (-1)^j 2u(\infty)) [\mathbb{A}^{-1} \nabla \Theta(W_0)]_g}{\Theta(\mathbf{f}_n + (-1)^j 2u(\infty)) i \vec{\tau}_1 \cdot \nabla \Theta(\mathbf{f}_n)}} \\ &\times \frac{\Theta(u_+(z) + (-1)^j u(\infty) + \mathbf{f}_n) r_+(z)}{\Theta(u_+(z) + (-1)^j u(\infty) + W_0)}, \quad j = 1, 2, \end{aligned} \quad (2.28)$$

where $z \in I$. It follows from Corollary 7.20, [BKT13], that for every $n \in \mathbb{N}$ we have $\Upsilon^{(1)}(z; \mathbf{f}_n) \equiv \pm \Upsilon^{(2)}(z; \mathbf{f}_n)$, where the choice of the sign depends on a particular n . It turns out that this sign is not essential, since the normalized singular functions $\hat{f}_n(z)$ and $\hat{h}_n(z)$, approximated through $\Upsilon^{(j)}(z; \mathbf{f}_n)$ (see below), are determined only up to a sign. Thus, we introduce $\Upsilon(z; \mathbf{f}_n)$ that, for a given $n \in \mathbb{N}$, coincides with both $\Upsilon^{(j)}(z; \mathbf{f}_n)$, $j = 1, 2$, modulo factor (-1) .

Now the asymptotics of singular functions is described by the following theorem.

Theorem 2.10. [Thm. 7.22] *The singular functions $\hat{f}_n(z)$ and $\hat{h}_n(z)$ of the system in (2.1) normalized in $L^2(I_i)$ and $L^2(I_e)$, respectively, are asymptotically given by*

$$\begin{aligned} \hat{f}_n(z) &= i \Im \left[2\Upsilon(z; \mathbf{f}_n) e^{-i \tilde{\kappa}_n \Im(\mathfrak{g}_+(z)) - i \Im(d_+(z))} \right] + \mathcal{O}(\tilde{\kappa}_n^{-1}), \quad z \in I_i, \\ \hat{h}_n(z) &= \Re \left[2\Upsilon(z; \mathbf{f}_n) e^{-i \tilde{\kappa}_n \Im(\mathfrak{g}_+(z)) - i \Im(d_+(z))} \right] + \mathcal{O}(\tilde{\kappa}_n^{-1}), \quad z \in I_e, \end{aligned} \quad (2.29)$$

where the approximation is uniform in any compact subset of the interior of I_i, I_e , respectively.

Corollary 2.11. [Cor. 7.24] *The singular functions $f_n(z)$ and $h_n(z)$ of the system (1.9) normalized in $L^2(I_i, \frac{1}{w(z)})$ and $L^2(I_e, \frac{1}{w(z)})$, respectively, are asymptotically given by*

$$\begin{aligned} \hat{f}_n(z) &= \sqrt{w(z)} \Im \left[2\Upsilon(z; \mathbf{f}_n) e^{-i \tilde{\kappa}_n \Im(\mathfrak{g}_+(z)) - i \Im(d_+(z))} \right] + \mathcal{O}(\tilde{\kappa}_n^{-1}), \quad z \in I_i, \\ \hat{h}_n(z) &= \sqrt{w(z)} \Re \left[2\Upsilon(z; \mathbf{f}_n) e^{-i \tilde{\kappa}_n \Im(\mathfrak{g}_+(z)) - i \Im(d_+(z))} \right] + \mathcal{O}(\tilde{\kappa}_n^{-1}), \quad z \in I_e, \end{aligned} \quad (2.30)$$

where the approximation is uniform in any compact subset of the interior of I_i, I_e , respectively.

3 Instability of the interior problem in Sobolev spaces

3.1 Continuation of f from I_i

The function $\psi(y)$ in (1.5) is analytic in $\mathbb{C} \setminus I_e$ and is known on I_i . If we can find the analytic continuation of $\psi(y)$ on (a_2, a_{2g+1}) , then, according to (1.6), we can solve the problem of reconstructing f on (a_2, a_{2g+1}) .

The idea of such reconstruction is straightforward. The eigenfunctions $\phi_n = \frac{1}{\sqrt{2}}(\hat{f}_n \chi_i + \hat{h}_n \chi_e)$ of the self-adjoint Hilbert-Schmidt integral operator $\hat{K} : L^2(I) \mapsto L^2(I)$ form an orthonormal basis in $L^2(I)$. Thus, \hat{f}_n, \hat{h}_n form orthonormal bases in $L^2(I_i), L^2(I_e)$ respectively, so that f_n, h_n form

orthonormal bases in the corresponding in $L^2(I_i, 1/w)$, $L^2(I_e, 1/w)$. Note that the former coincides with $L^2(I_i)$. Given $\psi \in L^2(I_i, 1/w)$ and $\varphi \in L^2(I_e, 1/w)$ we have

$$\psi = \sum \psi_n f_n \text{ on } I_i \text{ and } \varphi = \sum \varphi_n h_n \text{ on } I_e, \quad (3.1)$$

where $\sum \psi_n^2 < \infty$, $\sum \varphi_n^2 < \infty$. According to (2.1), $\mathcal{H}_e^{-1} h_n = 2\lambda_n f_n$, so that $\mathcal{H}_e^{-1} \varphi = \psi$ and (3.1) imply $\psi_n = 2\lambda_n \varphi_n$. In view of the asymptotics (1.12) of λ_n , we conclude that the coefficients ψ_n decay exponentially fast, so we have a very fast convergence of the series (3.1) for ψ . Note that, according to (1.9), the singular functions f_n are analytic in $\mathbb{C} \setminus I_e$. Thus the question of analytic continuation of ψ to (a_2, a_{2g+1}) through the series (3.1) is reduced to the question of convergence of $\psi = \sum \psi_n f_n$ in $(a_2, a_{2g+1}) \setminus I_i$.

Let \mathcal{I}_ω , $\omega > 0$, denote the set of all $z \in (a_2, a_{2g+1}) \setminus I_i$ that are at least ω away from the nearest branchpoint a_j , $j = 2, 3, \dots, 2g+1$. Below, we consider only such ω , that $a_j + \omega < a_{j+1} - \omega$ for all $j = 2, \dots, 2g$.

Lemma 3.1. *There exists a constant $C_\omega > 0$, such that for all $n \in \mathbb{N}$ and for all $z \in \mathcal{I}_\omega$*

$$|f_n(z)| \leq C_\omega e^{\varkappa_n(\Re g(z) + \frac{1}{2})}. \quad (3.2)$$

Lemma 3.1 follows from Lemma B.4, (2.12) and (2.1).

Lemma 3.2. *$|\Re g(z)| < \frac{1}{2}$ for any $z \in \mathbb{C} \setminus I$, with $\Re g(z) \equiv \frac{1}{2}$ on I_e and $\Re g(z) \equiv -\frac{1}{2}$ on I_i .*

Proof. Consider $g(z)$ on the main sheet of the Riemann surface \mathcal{R} with branchcuts on I . Note that $g(z)$ is Schwarz symmetrical and satisfies the jump conditions $g_+ + g_- \equiv 1$ on I_e and $g_+ + g_- \equiv -1$ on I_i , see Proposition 2.9. Thus, $\Re g(z) \equiv \frac{1}{2}$ on I_e and $\Re g(z) \equiv -\frac{1}{2}$ on I_i . The remaining statement follows from the maximal principle for harmonic functions. \square

Theorem 3.3. *For a given $\omega > 0$, the series $\psi(z) = \sum \psi_n f_n(z)$ converges absolutely and uniformly on \mathcal{I}_ω .*

Proof. Recall that $\lambda_n = \exp(-\varkappa_n)$. As a consequence of Lemma 3.1, we have

$$\sum |\psi_n f_n(z)| \leq 2C_\omega \varphi_* \sum e^{\varkappa_n(\Re g(z) - \frac{1}{2})}, \quad (3.3)$$

where $\varphi_* = \max_n \{|\varphi_n|\} < \infty$. In light of (1.12) and Lemma 3.2, the series in the right hand side of (3.3) converges absolutely and uniformly on \mathcal{I}_ω . \square

Corollary 3.4. *The series $\psi(z) = \sum \psi_n f_n(z)$ provides analytic continuation of ψ onto (a_2, a_{2g+1}) .*

Indeed, by choosing a sufficiently small ω , one can analytically continue $\psi(z)$ to any point in $(a_2, a_{2g+1}) \setminus I_i$ through this series.

3.2 Instability of analytic continuation in Sobolev norms

In the previous section we obtained a formula for analytic continuation of $\psi(y)$ from I_i to all of (a_2, a_{2g+1}) . Next we prove that analytic continuation of ψ from I_i is unstable for any pair of Sobolev spaces: $H^{s_1}(I_i) \rightarrow H^{-s_2}(\mathbf{J})$, where \mathbf{J} is any open set containing I_i . Clearly, it makes sense

to consider $s_1, s_2 > 0$. For simplicity we will assume that s_1 and s_2 are integers, so (see Chapter 1 in [ES97]):

$$\|f\|_{H^{s_1}(I_i)}^2 := \sum_{j=0}^{s_1} \int_{I_i} |f^{(j)}(y)|^2 dy, \quad (3.4)$$

and

$$\|f\|_{H^{-s_2}(J)} := \inf_{\tilde{f} \in H^{-s_2}(\mathbb{R}), f=\tilde{f}|_{\mathbf{J}}} \sup_{\phi \in C_0^\infty(\mathbb{R})} \frac{\left| \int_{\mathbb{R}} \tilde{f}(y) \overline{\phi(y)} dy \right|}{\|\phi\|_{H^{s_2}(\mathbb{R})}}. \quad (3.5)$$

Let γ be a collection of simple loops in the complex plane so that I_i is contained in the union of the interiors of the loops. We take γ to be sufficiently close to I_i . By the Cauchy integral theorem using the analyticity of f_n one can show that

$$\|f_n\|_{H^{s_1}(I_i)} \leq c(s_1, \gamma) \max_{z \in \gamma} |f_n(z)| \quad (3.6)$$

for some $c(s_1, \gamma) > 0$. Analogously to Lemma 3.1, it follows from Lemma B.4 that

$$\max_{z \in \gamma} |f_n(z)| \leq c_\gamma \exp(\kappa_n(\max_{z \in \gamma} \Re g(z) + \frac{1}{2})) \quad (3.7)$$

for some $c_\gamma > 0$. By taking γ sufficiently close to I_i , we can make $\max_{z \in \gamma} \Re g(z) + \frac{1}{2}$ as close to zero as we want.

Lemma 3.5. *One can find a sequence of intervals $J_n \subset \mathbf{J}$ with the following properties:*

1. *The length of each J_n is greater than a fixed positive constant independent of n ;*
2. *The distance of each J_n to I_i is greater than a fixed positive constant independent of n ; and*
3. *There exists $N > 0$ large enough such that*

$$|f_n(y)| \geq c \exp(\kappa_n(\Re g(y) + \frac{1}{2})), \quad n \geq N, \quad y \in J_n, \quad (3.8)$$

for some $c > 0$ independent of n .

Lemma 3.5 is proven in Appendix B. By property 1 in Lemma 3.5 we can find $L > 0$ such that the length of each interval J_n is greater than or equal to L . Then we select a real-valued function $\phi \in C_0^\infty([-L/2, L/2])$, $\phi \geq 0$, $\phi \not\equiv 0$. By shifting ϕ appropriately, we get a collection of functions $\phi_n \in C_0^\infty(J_n)$ and they all have the same $H^{s_2}(\mathbb{R})$ -norm. Using the facts that: (i) f and \tilde{f} coincide on \mathbf{J} (cf. (3.5)); (ii) f_n 's are real-valued on \mathbf{J} , and; (iii) f_n 's do not change sign on J_n for n large (cf. (3.8)), equation (3.5) immediately yields

$$\|f_n\|_{H^{-s_2}(\mathbf{J})} \geq c_\phi \min_{y \in J_n} |f_n(y)|, \quad n \geq N, \quad (3.9)$$

for some $c_\phi > 0$.

From the second property in Lemma 3.5, by choosing γ sufficiently close to I_i so that all J_n are in the exterior of γ and $\text{dist}(\gamma, \cup_n J_n) > 0$, we get $\inf_{y \in \cup_n J_n} \Re g(y) > \max_{z \in \gamma} \Re g(z)$. Hence,

$$\frac{\exp(\kappa_n(\min_{y \in J_n} \Re g(y) + \frac{1}{2}))}{\exp(\kappa_n(\max_{z \in \gamma} \Re g(z) + \frac{1}{2}))} \rightarrow \infty, \quad n \rightarrow \infty. \quad (3.10)$$

Hence it follows from (3.7) and (3.8) that the quantity $\|f_n\|_{H^{-s_2}(\mathbf{J})}$ cannot be bounded in terms of $\|f_n\|_{H^{s_1}(I_i)}$. Since the Sobolev norm $\|f\|_{H^s}$ is a monotonically increasing function of s (provided that f belongs to the appropriate spaces), our argument proves the following result.

Theorem 3.6. *Fix an open set $\mathbf{J} \supset I_i$. The operation of analytic continuation from I_i to \mathbf{J} described in Corollary 3.4 cannot be extended to a continuous operator $H^{s_1}(I_i) \rightarrow H^{-s_2}(\mathbf{J})$ for any s_1, s_2 .*

Theorem 3.6 shows that analytic continuation is more unstable than calculation of any number of derivatives. An interesting question is to estimate the degree of ill-posedness of analytic continuation. This can be done, for example, by finding a Hilbert space \mathcal{A} on which the operator of analytic continuation is bounded. It is clear that the space \mathcal{A} should contain at least all functions in the range of $\mathcal{H}_e^{-1} : L^2(I_e, 1/w) \rightarrow L^2(I_i, 1/w)$. If $\psi \in \mathcal{A}$, but ψ is not in the range of \mathcal{H}_e^{-1} , then the analytic continuation of ψ is understood via the summation of the series in Corollary 3.4.

Let w_n be a sequence of positive numbers. Introduce the following space:

$$\mathcal{A} := \{\psi \in L^2(I_i) : \sum_{n \geq 0} w_n^2 |\psi_n|^2 < \infty\}, \quad (3.11)$$

where

$$\psi_n := \langle \psi, f_n \rangle := \int_{I_i} \psi(y) f_n(y) \frac{1}{w(y)} dy. \quad (3.12)$$

It is obvious that \mathcal{A} is a Hilbert space with the inner product defined by the formula

$$\langle \psi^{(1)}, \psi^{(2)} \rangle := \sum_{n \geq 0} w_n^2 \psi_n^{(1)} \overline{\psi_n^{(2)}}. \quad (3.13)$$

Theorem 3.7. *Fix an open set \mathbf{J} , whose closure is a subset of (a_2, a_{2g+1}) . Suppose that each connected component of \mathbf{J} contains at least one of the intervals that make up I_i . Suppose the sequence of w_n 's is such that the limit below exists and satisfies*

$$0 < \lim_{n \rightarrow \infty} \left\{ \frac{w_n}{n} \exp(-\varkappa_n (\sup_{z \in \mathbf{J}} \Re g(z) + \frac{1}{2})) \right\} < \infty. \quad (3.14)$$

Then one has: (1) $\mathcal{H}_e^{-1}(L^2(I_e, 1/w)) \subset \mathcal{A}$, and; (2) the operator of analytic continuation acting between the spaces $\mathcal{A} \rightarrow L^2(\mathbf{J})$ is continuous.

Proof. Similarly to the proof of Theorem 3.3, it is easily seen that assertion (1) holds. Now we prove assertion (2). First we show that

$$\max_{z \in \mathbf{J}} |f_n(z)| \leq c_{\mathbf{J}} \exp(\varkappa_n (\sup_{z \in \mathbf{J}} \Re g(z) + \frac{1}{2})) \quad (3.15)$$

for some $c_{\mathbf{J}} > 0$. Denote $G := \sup_{z \in \mathbf{J}} \Re g(z)$. Let γ be a collection of simple contours in \mathbb{C} containing the components of $\mathbf{J} \cap I_i$ in their interior. By making γ as close to these component as we need and using Lemma 3.2, we can find γ such that $\sup_{z \in \gamma} \Re g(z) < G$. Now (3.15) follows

immediately by using inequalities (3.2) and (3.7) combined with the maximum modulus principle. Finally, to prove (2) we fix any $N > 0$. Then

$$\int_J \left| \sum_{n=0}^N \psi_n f_n(z) \right|^2 dz \leq |J| \left(\sum_{n=0}^N |\psi_n| \sup_{z \in J} |f_n(z)| \right)^2 \leq c \left(\sum_{n=0}^N |\psi_n| \frac{w_n}{n} \right)^2 \leq c \sum_{n=0}^N (|\psi_n| w_n)^2 \sum_{n=0}^N \frac{1}{n^2}, \quad (3.16)$$

where $c > 0$ is some constant. By taking the limit $N \rightarrow \infty$ the desired assertion follows immediately. \square

Remark 3.8. *Using the fact that the singular functions f_n are analytic on \mathbf{J} and the coefficients ψ_n go to zero sufficiently fast, similarly to the proof of Theorem 3.3 and (3.16) it is easy to see that each $\psi \in \mathcal{A}$ defined on \mathbf{J} via the series in Corollary 3.4 is a uniform limit of analytic functions. Hence the continuation of ψ from I_i to \mathbf{J} via the series and via the conventional analytic continuation coincide.*

4 Approximation in $L^2(I_i)$

According to Theorem 2.10, the normalized singular functions \widehat{f}_n are approximated by

$$\tilde{f}_n := i\Im \left[2\Upsilon(z; \mathbf{f}_n) e^{-i\kappa_n \Im(\mathfrak{g}_+(z)) - i\Im(d_+(z))} \right] \quad (4.1)$$

with accuracy $O(n^{-1})$ in the sup-norm (uniformly) on any compact subset of the interior of I_i . In this subsection we discuss this approximation in $L^2(I_i)$. We will use $\|f\|$ to denote the L^2 norm of $f \in L^2(I_i)$.

Lemma 4.1. *Let ω_0 be so small that each interval $(a_k - \omega_0, a_k + \omega_0)$ contains no endpoints except a_k . Then there exists some $\eta > 0$, such that*

$$\forall k \in \{3, \dots, 2g\}, \forall n \in \mathbb{N}, \forall \omega \in (0, \omega_0) : \quad |\tilde{f}_n(z)| \leq \frac{\eta}{|z - a_k|^{\frac{1}{4}}} \quad \text{on } (a_k - \omega, a_k + \omega). \quad (4.2)$$

Lemma 4.1 follows from Lemma B.2 and Corollary A.19.

Lemma 4.2. *The norms of $\tilde{f}_n(z)$ (4.1) satisfy the asymptotic expansion*

$$\|\tilde{f}_n\| = 1 + O(n^{-1}), \quad n \rightarrow \infty. \quad (4.3)$$

The proof of this lemma can be found in Appendix B.

Let $\omega > 0$ and define $I_i^\omega = I_i \setminus \bigcup_{k=3}^{2g} (a_k - \omega, a_k + \omega)$. If $f \in L^2(I_i)$, then $\|f\|^2 = \|f\|_b^2 + \|f\|_t^2$, where $\|f\|_b$ denotes the norm of f in $L^2(I_i^\omega)$ (in the bulk) and $\|f\|_t$ denotes the norm of f in $L^2(I_i \setminus I_i^\omega)$ (in the tails).

According to Theorem 2.10, for any $\omega \in (0, \omega_0)$ there exists some $P_\omega > 0$, such that

$$\|\widehat{f}_n - \tilde{f}_n\|_b \leq \frac{P_\omega}{n}. \quad (4.4)$$

Theorem 4.3. \tilde{f}_n approximate \hat{f}_n in $L^2(I_i)$, that is, $\forall \epsilon > 0 \exists n_0 \in \mathbb{N}$ such that $\forall n > n_0$: $\|\hat{f}_n - \tilde{f}_n\| < \epsilon$.

Proof. According to (4.2), $\|\tilde{f}_n\|_t \leq 2\sqrt{g-1}\eta\omega^{\frac{1}{4}}$ for all $n \in \mathbb{N}$. As implied by Lemma 4.2, there exist some $Q_\omega > 0$, such that $\|\tilde{f}_n\| \geq 1 - \frac{Q_\omega}{n}$. Since $1 - \frac{Q_\omega}{n} \leq \|\tilde{f}_n\| \leq \|\tilde{f}_n\|_b + \|\tilde{f}_n\|_t$, we obtain $\|\tilde{f}_n\|_b \geq 1 - 2\sqrt{g-1}\eta\omega^{\frac{1}{4}} - \frac{Q_\omega}{n}$. Then, according to (4.4),

$$\begin{aligned} \|\hat{f}_n\|_b &\geq \|\tilde{f}_n\|_b - \|\hat{f}_n - \tilde{f}_n\|_b \geq 1 - 2\sqrt{g-1}\eta\omega^{\frac{1}{4}} - \frac{P_\omega}{n} - \frac{Q_\omega}{n}, \quad \text{so that} \\ \|\hat{f}_n\|_t^2 &= 1 - \|\hat{f}_n\|_b^2 \leq 2\left(2\sqrt{g-1}\eta\omega^{\frac{1}{4}} + \frac{P_\omega + Q_\omega}{n}\right). \end{aligned} \quad (4.5)$$

Thus,

$$\|\hat{f}_n - \tilde{f}_n\| \leq \|\hat{f}_n - \tilde{f}_n\|_b + \|\hat{f}_n\|_t + \|\tilde{f}_n\|_t \leq 2\sqrt{g-1}\eta\omega^{\frac{1}{4}} + \frac{P_\omega + Q_\omega}{n} + \sqrt{2\left(2\sqrt{g-1}\eta\omega^{\frac{1}{4}} + \frac{P_\omega + Q_\omega}{n}\right)}. \quad (4.6)$$

It is clear that for a small ϵ condition $2\sqrt{g-1}\eta\omega^{\frac{1}{4}} + \frac{P_\omega + Q_\omega}{n} < \frac{\epsilon^2}{4}$ would imply $\|\hat{f}_n - \tilde{f}_n\| \leq \epsilon$. Choose $\omega^{\frac{1}{4}} = \frac{\epsilon^2}{16\sqrt{g-1}\eta}$. Then the former inequality holds for all $n > \frac{8(P_\omega + Q_\omega)}{\epsilon^2}$. The proof is completed. \square

A Approximate solution of the RHP 2.2 and related results from [BKT13]

Construction of the leading order approximation of the solution $\Gamma(z; \lambda)$ of the RHP 2.2 in the limit $\lambda \rightarrow 0^+$ is at the heart of our method. We also have to control the accuracy of such approximation. We employ the nonlinear steepest descent method of Deift and Zhou, that allows to asymptotically reduce the original RHP (RHP 2.2) to a certain RHP with constant jumps (RHP A.5) that one can solve explicitly. The asymptotic reduction consists of a sequence of transformations of the RHP 2.2, some of them equivalent and some asymptotic (with the error estimates for the later). The key idea is a factorization of the jump matrix with a subsequent contour deformation, where each factor “acquires” its own jump-contour in the process. In this appendix we only briefly outline some main points of the reduction of the RHP 2.2 and provide a solution to the corresponding “reduced” RHP with constant jumps. The details can be found in [BKT13]. There exists a large and rapidly growing literature about the method Deift and Zhou and its various applications, see, for example, [Dei99], [BKL+08]. We also include some facts about theta divisors as well as some further results from [BKT13] that are used in the proof of technical lemmas in Appendix B.

Let Σ be an oriented collection of contours that partition \mathbb{C} into a finite number of open regions and let $V(z)$ be an $n \times n$ matrix valued function defined on Σ , satisfying certain conditions at the nodes of Σ .⁴ A (somewhat) general formulation of a matrix RHP can be stated as follows. We do not get here into the details of the smoothness of Σ and $V(z)$.

Riemann-Hilbert Problem A.1. Find an $n \times n$ matrix-function $M(z)$ that:

⁴A point $z \in \Sigma$ is called a node if three or more branches of Σ emanates from z . We assume Σ has no more than finitely many nodes.

- is analytic in each element of partition, induced by the contour Σ ;
- for any $z \in \Sigma$ that is not a node $M(z)$ admits non-tangential boundary values $M_{\pm}(z)$ from the corresponding sides of Σ and

$$M_+(z) = M_-(z)V(z); \quad (\text{A.1})$$

•

$$\lim_{z \rightarrow \infty} M(z) = \mathbf{1}. \quad (\text{A.2})$$

In general, the existence of a solution to the RHP (A.1) is not guaranteed. The nonlinear steepest descent method is based upon the following “small norm theorem”.

Theorem A.2. *Let N_p denotes the norms of $V(z) - \mathbf{1}$ in $L^p(\Sigma, dz)$. Then*

- There is a constant C_{Σ} such that if $N_{\infty} < C_{\Sigma}^{-1}$ the solution of the RHP A.1 exists;
- In this case

$$\|M(z) - \mathbf{1}\| \leq \frac{1}{2\pi \text{dist}(z, \Sigma)} \left(N_1 + \frac{C_{\Sigma} N_2^2}{1 - C_{\Sigma} N_{\infty}} \right) \quad (\text{A.3})$$

for every $z \in \mathbb{C} \setminus \Sigma$.

The name of this theorem reflects the fact that the solution $M(z)$ of the RHP A.1 is close (pointwise) to the identity matrix $\mathbf{1}$ if the norms $N_{1,2}$ are small.

Let $\varkappa = -\ln \lambda$. Then $\varkappa > 0$ when $\lambda \in (0, 1)$ and $\varkappa \rightarrow \infty$ as $\lambda \rightarrow 0$. The first transformation is replacing $\Gamma(z; \lambda)$ with $Y(z; \varkappa)$ by

$$Y(z; \varkappa) = e^{-(\varkappa \mathbf{g}(\infty) + d(\infty) \sigma_3)} \Gamma(z; e^{-\varkappa}) e^{(\varkappa \mathbf{g}(z) + d(z)) \sigma_3}, \quad (\text{A.4})$$

where $\mathbf{g}(z), d(z)$ are defined by (2.22) and the Pauli matrices are defined as

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then direct calculations show that the RHP 2.2 for $\Gamma(z; \lambda)$ is reduced to the following equivalent RHP for Y .

Riemann-Hilbert Problem A.3. *Find a 2×2 matrix-function $Y(z; \varkappa)$ with the following properties:*

- (a) $Y(z; \varkappa)$ is analytic in $\mathbb{C} \setminus [a_1, a_{2g+2}]$;
- (b) $Y(z; \varkappa)$ satisfies the jump conditions

$$\begin{aligned} Y_+ &= Y_- \begin{bmatrix} e^{(\varkappa \mathbf{g} + d)_+ - (\varkappa \mathbf{g} + d)_-} & 0 \\ i w e^{\varkappa(\mathbf{g}_+ + \mathbf{g}_- + 1) + d_+ + d_-} & e^{-(\varkappa \mathbf{g} + d)_+ + (\varkappa \mathbf{g} + d)_-} \end{bmatrix}, \quad z \in I_i, \\ Y_+ &= Y_- \begin{bmatrix} e^{(\varkappa \mathbf{g} + d)_+ - (\varkappa \mathbf{g} + d)_-} & -\frac{i}{w} e^{-\varkappa(\mathbf{g}_+ + \mathbf{g}_- - 1) - d_+ - d_-} \\ 0 & e^{-(\varkappa \mathbf{g} + d)_+ + (\varkappa \mathbf{g} + d)_-} \end{bmatrix}, \quad z \in I_e \\ \text{and } Y_+ &= Y_- e^{[(\varkappa \mathbf{g} + d)_+ - (\varkappa \mathbf{g} + d)_-] \sigma_3} \quad \text{on } [a_{2j}, a_{2j+1}], \quad j = 1, \dots, g; \end{aligned} \quad (\text{A.5})$$

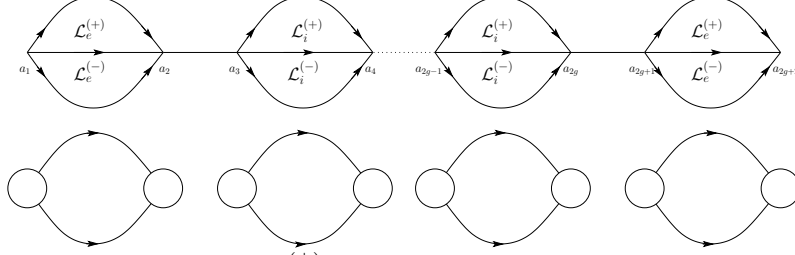


Figure 3: The regions of the lenses $\mathcal{L}_{i,e}^{(\pm)}$ (above) and the jumps of the error matrix \mathcal{E} (below).

(c) $Y = \mathbf{1} + O(z^{-1})$ as $z \rightarrow \infty$, and;

(d) Near the branchpoints (we indicate the behavior for the columns if these have different behaviors)

$$Y(z; \varkappa) = [\mathcal{O}(1), \mathcal{O}(z - a_j)^{-\frac{1}{2}}], \quad j = 1, 2g+2; \quad Y(z; \varkappa) = O(\ln(z - a_j)), \quad j = 2, \dots, 2g+1. \quad (\text{A.6})$$

In the next transformation (now of the RHP A.3) we first factorize the triangular jump matrices in (A.5) as

$$\begin{aligned} Y_+ &= Y_- \begin{bmatrix} 1 & \frac{e^{-\varkappa(2\mathfrak{g}_- + 1) - 2d_-}}{iw} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{e^{-\varkappa(2\mathfrak{g}_+ + 1) - 2d_+}}{iw} \\ 0 & 1 \end{bmatrix} \text{ on } I_i, \\ Y_+ &= Y_- \begin{bmatrix} 1 & 0 \\ iwe^{\varkappa(2\mathfrak{g}_- - 1) + 2d_-} & 1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ iwe^{\varkappa(2\mathfrak{g}_+ - 1) + 2d_+} & 1 \end{bmatrix} \text{ on } I_e, \end{aligned} \quad (\text{A.7})$$

and then put each of the three factors (for I_e and for I_i) on its own jump contour as described below.

The validity of the factorization can be checked directly, taking into the account the identities $-\varkappa(\mathfrak{g}_+ + \mathfrak{g}_- \pm 1) - \ln w - d_+ - d_- \equiv 0$ that hold on I_i and I_e respectively, see (2.24), (2.26). The left and right (triangular) matrices in both factorizations (on I_i and on I_e) admit analytic extension on the left/right vicinities of the corresponding segments because they are boundary values of analytic matrices in those vicinities. This suggests opening of the lenses $\partial\mathcal{L}_e^{(\pm)}$, $\partial\mathcal{L}_i^{(\pm)}$ around the corresponding intervals of $I_e \cup I_i$, see Figure 3 upper panel, and introduction of the new unknown matrix

$$Z(z; \varkappa) = \begin{cases} Y(z; \varkappa) & \text{outside the lenses,} \\ Y(z; \varkappa) \begin{bmatrix} 1 & 0 \\ \mp iwe^{\varkappa(2\mathfrak{g}-1)+2d} & 1 \end{bmatrix} & z \in \mathcal{L}_e^{(\pm)}, \\ Y(z; \varkappa) \begin{bmatrix} 1 & \mp \frac{1}{iw} e^{-\varkappa(2\mathfrak{g}+1)-2d} \\ 0 & 1 \end{bmatrix} & z \in \mathcal{L}_i^{(\pm)}. \end{cases} \quad (\text{A.8})$$

Consequently, after the second (equivalent) transformation we obtain the following RHP for the matrix Z .

Riemann-Hilbert Problem A.4. Find the matrix Z , analytic on the complement of the arcs of Figure 3, satisfying the jump conditions (note also the orientations marked in Figure 3)

$$Z_+(z; \kappa) = Z_-(z; \kappa) \begin{cases} e^{i(\kappa\Omega_{\mu(j)} + \delta_{\mu(j)})\sigma_3} & z \in [a_{2j}, a_{2j+1}], \quad j = 1, \dots, g, \\ \begin{bmatrix} 1 & 0 \\ iw e^{\kappa(2g-1)+2d} & 1 \end{bmatrix} & z \in \partial\mathcal{L}_e^{(\pm)} \setminus \mathbb{R}, \\ \begin{bmatrix} 1 & \frac{1}{iw} e^{-\kappa(2g+1)-2d} \\ 0 & 1 \end{bmatrix} & z \in \partial\mathcal{L}_i^{(\pm)} \setminus \mathbb{R}, \\ i\sigma_1 & z \in I_i, \\ -i\sigma_1 & z \in I_e, \end{cases} \quad (\text{A.9})$$

normalized by

$$Z(z; \kappa) \rightarrow \mathbf{1}, \quad z \rightarrow \infty, \quad (\text{A.10})$$

and with the same endpoint behavior as Y near the endpoints a_j 's, see (A.6). Here

$$\vec{\delta} = [\delta_1, \dots, \delta_{g-1}, \delta_0]^t = 2\pi L^{-1}((2u(\infty) - u(a_{2g+2})), \quad \vec{\Omega} = [\Omega_1, \dots, \Omega_{g-1}, \Omega_0]^t = -2iL^{-1}\tau_1$$

and $\mu(g) = 0$, $\mu(j) = j$ for all $j \neq g$, see Proposition 2.9.

In the third and final transformation we would like to (asymptotically) reduce the RHP A.4 for $Z(z; \kappa)$ to the following RHP for $\Psi = \Psi(z; \kappa)$.

Riemann-Hilbert Problem A.5 (Model problem). Find a matrix $\Psi = \Psi(z; \kappa)$, analytic on $\mathbb{C} \setminus [a_1, a_{2g+2}]$ and satisfying the following conditions:

$$\begin{aligned} \Psi_+ &= \Psi_-(-1)^{s(j)}(i\sigma_1), & z \in [a_{2j-1}, a_{2j}], \quad j = 1, \dots, g+1, \\ \Psi_+ &= \Psi_- e^{i(\kappa\Omega_{\mu(j)} + \delta_{\mu(j)})\sigma_3} & z \in [a_{2j}, a_{2j+1}], \quad j = 1, \dots, g, \\ \Psi(z) &= \mathcal{O}(|z - a_j|^{-\frac{1}{4}}), & z \rightarrow a_j, \quad j = 1, \dots, 2g+2, \\ \Psi(z) &= \mathbf{1} + \mathcal{O}(z^{-1}), & z \rightarrow \infty, \quad \text{and} \quad \Psi_{\pm}(z) \in L^2([a_1, a_{2g+2}]). \end{aligned} \quad (\text{A.11})$$

Here $s(j) = \delta_{j,1} + \delta_{j,g+1}$, where $\delta_{j,k}$ denotes the Kronecker delta.

The RHP A.5 is not equivalent to the RHP A.4 since the former does not have jumps on the lenses $\partial\mathcal{L}_e^{(\pm)} \setminus \mathbb{R}$, $\partial\mathcal{L}_i^{(\pm)} \setminus \mathbb{R}$. Also, a different behavior is required near the branchpoints a_j , $j = 1, 2, \dots, 2g+2$. However, as a consequence of Theorem A.2, the solution $\Psi(z; \kappa)$ of the RHP A.5 will approximate the solution $Z(z; \kappa)$ of the RHP A.4, if the jump matrices on the lenses $\partial\mathcal{L}_e^{(\pm)} \setminus \mathbb{R}$, $\partial\mathcal{L}_i^{(\pm)} \setminus \mathbb{R}$ in (A.9) will be small in the norms N_1, N_2 and N_{∞} . The choice of the so-called g -function $\mathfrak{g}(z)$ in the transformation (A.4), as well as of the lenses $\mathcal{L}_{e,i}^{(\pm)}$, is defined by the requirements that

$$\Re(2\mathfrak{g} - 1) < 0 \quad \text{on} \quad \partial\mathcal{L}_e^{(\pm)} \setminus \mathbb{R} \quad \text{and} \quad \Re(2\mathfrak{g} + 1) > 0 \quad \text{on} \quad \partial\mathcal{L}_i^{(\pm)} \setminus \mathbb{R}. \quad (\text{A.12})$$

If these requirements hold, the jump matrices (A.9) on the contours $\partial\mathcal{L}_e^{(\pm)} \setminus \mathbb{R}$, $\partial\mathcal{L}_i^{(\pm)} \setminus \mathbb{R}$ approach $\mathbf{1}$ exponentially fast as $\kappa \rightarrow \infty$ in any L^p , $p < \infty$ but *not* in L^{∞} , because this convergence is uniform away from small vicinities of the branchpoints. These vicinities which require special consideration. Namely, to match RHP A.9 with RHP A.5, we need to construct special *local parametrices* in these vicinities of the branchpoints.

The discrepancy between $Z(z; \varkappa)$ and $\Psi(z; \varkappa)$, the latter modified by the parametrices near the branchpoints, is represented by the so-called error matrix $\mathcal{E}(z; \varkappa)$. The error matrix also satisfies a certain RHP; the jump contours of this RHP are shown on Figure 3. We already know that the jump matrices on the arcs $\partial\mathcal{L}_e^{(\pm)} \setminus \mathbb{R}, \partial\mathcal{L}_i^{(\pm)} \setminus \mathbb{R}$ away from the branchpoints should approach $\mathbf{1}$ exponentially fast in $\varkappa \rightarrow \infty$. The parametrices, constructed in [BKT13], ensure that the jump matrices on the circles, shown on Figure 3, behave like $\mathbf{1} + O(\varkappa^{-1})$ as $\varkappa \rightarrow \infty$ in any L^p , including L^∞ . Thus, according to Theorem A.2, $\Psi(z; \varkappa)$ is $O(\varkappa^{-1})$ close to $Z(z; \varkappa)$ uniformly in z outside small circles around the branchpoints.

Now, by reversing the chain of transformations, one obtains the following summary of the steepest descent analysis of [BKT13]: let constants $\epsilon, \rho_z, \rho_0 > 0$ be fixed and sufficiently small. Then

$$\Gamma(z; e^{-\varkappa}) = e^{(\varkappa g(\infty) + d(\infty))\sigma_3} \mathcal{E}(z; \varkappa) \Psi(z; \varkappa) \begin{bmatrix} 1 & \frac{\pm 1}{iw} e^{-\varkappa(2g+1)-2d} \widehat{\chi}_{i,\pm} \\ \pm iwe^{\varkappa(2g-1)+2d} \widehat{\chi}_{e,\pm} & 1 \end{bmatrix} e^{-(\varkappa g(z) + d(z))\sigma_3}, \quad (\text{A.13})$$

where $\widehat{\chi}_e^\pm, \widehat{\chi}_i^\pm$ are the characteristic (indicator) functions of the sets $\mathcal{L}_e^{(\pm)}, \mathcal{L}_i^{(\pm)}$ respectively, see Figure 3, and

$$\mathcal{E}(z; \varkappa) = \mathbf{1} + \frac{\mathcal{O}(\varkappa^{-1})}{1 + |z|} \quad (\text{A.14})$$

uniformly in the domain

$$\Im \varkappa < \epsilon, \quad |\Theta(W(\varkappa) - W_0)| > \rho_0, \quad |z - a_j| > \rho_z, \quad j = 1, \dots, 2g + 2. \quad (\text{A.15})$$

The matrix $\mathcal{E}(z; \varkappa)$ solves an auxiliary RHP where all the jumps satisfy the assumption of Theorem A.2; in particular it is important for us that \mathcal{E} does not have a jump on the main arcs $I_e \cup I_i$. This implies that the following matrix

$$\Gamma^{(\infty)}(z; e^{-\varkappa}) = e^{(\varkappa g(\infty) + d(\infty))\sigma_3} \Psi(z; \varkappa) \begin{bmatrix} 1 & \frac{\pm 1}{iw} e^{-\varkappa(2g+1)-2d} \chi_{i,\pm} \\ \pm iwe^{\varkappa(2g-1)+2d} \chi_{e,\pm} & 1 \end{bmatrix} e^{-(\varkappa g(z) + d(z))\sigma_3} \quad (\text{A.16})$$

has the exact same jumps as $\Gamma(z; \lambda)$ on $I_i \cup I_e$.

In terms of the Riemann Theta functions Θ , the explicit solution to the RHP A.5 is given by

$$\Psi(z; \varkappa) = C_0 \begin{bmatrix} \frac{\Theta(u(z) - u(\infty) - W_0 + W)r(z)}{\Theta(W - W_0)\Theta(u(z) - u(\infty) - W_0)} & \frac{\Theta(-u(z) - u(\infty) - W_0 + W)r(z)}{\Theta(W - W_0)\Theta(-u(z) - u(\infty) - W_0)} \\ \frac{-\Theta(u(z) + u(\infty) - W_0 + W)r(z)}{\Theta(W - W_0)\Theta(u(z) + u(\infty) - W_0)} & \frac{-\Theta(-u(z) + u(\infty) - W_0 + W)r(z)}{\Theta(W - W_0)\Theta(-u(z) + u(\infty) - W_0)} \end{bmatrix} \quad (\text{A.17})$$

where vectors $W = W(\varkappa)$ and W_0 are defined in (2.22) and $C_0 = [\mathbb{A}^{-1} \nabla \Theta(W_0)]_g$ is the last entry of the vector $\mathbb{A}^{-1} \nabla \Theta(W_0)$. The constant $C_0 \neq 0$.

Lemma A.6. *The endpoints a_n , $n \in J$, and infinity (on one of the sheets) are the only zeroes (in z) of the functions $\Theta(-(-1)^k u(z) + (-1)^j u(\infty) + W_0)$, $j, k = 1, 2$. All these zeroes are simple.*

According to Lemma A.6, $\Psi(z; \varkappa)$ is well defined if the denominator $\Theta(W - W_0) \neq 0$.

Theorem A.7. [Thm. 5.3, [BKT13]] *The RHP A.5 has a solution if and only if $\Theta(W - W_0) \neq 0$.*

As a consequence, we obtained the condition (2.20) for the logarithms of approximate eigenvalues. According to (2.11), in order to approximate singular functions, we need to calculate the residues of (A.17).

Proposition A.8 (Symmetry). *If $\Psi(z; \varkappa)$ satisfies the RHP A.5 then $\det \Psi \equiv 1$ and $\tilde{\Psi}(z) \equiv \Psi(z)$, where $\tilde{\Psi}(z; \varkappa) = \overline{\Psi(\bar{z}; \bar{\varkappa})}$. In particular, for $\varkappa \in \mathbb{R}$, $\Psi_{j1+}(z; \varkappa) = \overline{\Psi_{j1-}(z; \varkappa)}$ for any $z \in I = I_i \cup I_e$.*

Further analysis of singular functions requires some information about zeroes of the Theta function, given in Section A.1

A.1 Theta divisors and some related results from [BKT13]

Definition A.9. *Let a_1 be a base-point of the Abel map $\mathbf{u}(z)$ (see (2.19)) on the hyperelliptic Riemann surface \mathcal{R} of $\sqrt{\prod_{j=1}^{2g+2} (z - a_j)}$. Then the vector of Riemann constants \mathcal{K} is*

$$\mathcal{K} = \sum_{j=1}^g \mathbf{u}(a_{2j+1}). \quad (\text{A.18})$$

Theorem A.10 ([FK92], p. 308). *Let $\mathbf{f} \in \mathbb{C}^g$ be arbitrary, and denote by $\mathbf{u}(p)$ the Abel map extended to the whole Riemann surface. The (multi-valued) function $\Theta(\mathbf{u}(z) - \mathbf{f})$ on the Riemann surface either vanishes identically or vanishes at g points p_1, \dots, p_g (counted with multiplicity). In the latter case we have*

$$\mathbf{f} = \sum_{j=1}^g \mathbf{u}(p_j) + \mathcal{K}. \quad (\text{A.19})$$

Remark A.11. *Description of the vectors \mathbf{f} that lead to identically vanishing $\Theta(\mathbf{u}(z) - \mathbf{f})$ is more involved and will not be discussed here.*

Let us denote by $\Lambda_\tau = \mathbb{Z}^g + \tau \mathbb{Z}^g \subset \mathbb{C}^g$ the lattice of periods. The **Jacobian** is the quotient $\mathbb{J}_\tau = \mathbb{C}^g \bmod \Lambda_\tau$ and it is a compact torus of real dimension $2g$ on account of Theorem 2.4.

Definition A.12. *The **theta divisor** is the locus $\mathbf{e} \in \mathbb{J}_\tau$ such that $\Theta(\mathbf{e}) = 0$. It will be denoted by the symbol (Θ) .*

Proposition A.13 (Prop.7.1, [BKT13]). *If $W \in \mathbb{R}^g$ and W_0 is given as in (2.18), then*

$$\Theta(W - W_0) = 0 \iff W = \sum_{\ell=1}^{g-1} (\mathbf{u}(p_{\ell+1}) - \mathbf{u}(a_{j_\ell})) \bmod \mathbb{Z}^g, \quad (\text{A.20})$$

where $p_{\ell+1} = (z_{\ell+1}, R_{\ell+1})$, $\ell = 1, \dots, g-1$, are arbitrary points with $z_{\ell+1} \in [a_{2\ell}, a_{2\ell+1}]$, $\ell = 1, \dots, g-2$, and $z_g \in \mathbb{R} \setminus [a_1, a_{2g+2}]$ (i.e. belonging to the cycles $A_{1+\ell}$, $\ell = 1, \dots, g-1$), and $j_\ell \in J = \{1, 5, 7, 9, 11, \dots, 2g-1\}$.

Remark A.14. *Proposition A.13 explicitly parametrizes the hypersurface $\Theta(W - W_0) = 0$, $W \in \mathbb{R}^g$ in terms of $g-1$ points p_2, \dots, p_g belonging to the cycles A_2, \dots, A_g . For the special values $\varkappa = \tilde{\varkappa}_n$, when the line $W(\varkappa)$ (given by (2.18)) intersects with this hypersurface, we shall denote the corresponding points on the cycles A_2, \dots, A_g by $p_2^{(n)}, \dots, p_g^{(n)}$ with $\vec{p}_n = (p_2^{(n)}, \dots, p_g^{(n)})$. According*

to (2.22) and Theorem A.10, $\mathbf{f}_n = \sum_{j=2}^g \mathbf{u}(p_j^{(n)}) + \mathcal{K}$. For this reason it makes sense to consider $\mathbf{f}(\vec{p}) := \sum_{j=2}^g \mathbf{u}(p_j) + \mathcal{K}$, where $\vec{p} = (p_2, \dots, p_g)$, as a function on the (universal cover) of the torus $A_2 \times \dots \times A_g$. Then we have $\mathbf{f}_n = \mathbf{f}(\vec{p}_n)$.

Lemma A.15 (Lem. 7.14, [BKT13]). **(1)** For $\Psi(z; \varkappa)$ from (A.17) we have

$$\operatorname{res}_{\varkappa=\tilde{\varkappa}_n} \Psi(z; \varkappa) = C_0 \begin{bmatrix} \frac{i\pi\Theta(\mathbf{u}(z) - \mathbf{u}(\infty) + \mathbf{f}(\vec{p}_n))r(z)}{\tilde{\tau}_1 \cdot \nabla\Theta(\mathbf{f}(\vec{p}_n))\Theta(\mathbf{u}(z) - \mathbf{u}(\infty) + W_0)} & \frac{i\pi\Theta(-\mathbf{u}(z) - \mathbf{u}(\infty) + \mathbf{f}(\vec{p}_n))r(z)}{\tilde{\tau}_1 \cdot \nabla\Theta(\mathbf{f}(\vec{p}_n))\Theta(-\mathbf{u}(z) - \mathbf{u}(\infty) + W_0)} \\ \frac{-i\pi\Theta(\mathbf{u}(z) + \mathbf{u}(\infty) + \mathbf{f}(\vec{p}_n))r(z)}{\tilde{\tau}_1 \cdot \nabla\Theta(\mathbf{f}(\vec{p}_n))\Theta(\mathbf{u}(z) + \mathbf{u}(\infty) + W_0)} & \frac{-i\pi\Theta(-\mathbf{u}(z) + \mathbf{u}(\infty) + \mathbf{f}(\vec{p}_n))r(z)}{\tilde{\tau}_1 \cdot \nabla\Theta(\mathbf{f}(\vec{p}_n))\Theta(-\mathbf{u}(z) + \mathbf{u}(\infty) + W_0)} \end{bmatrix}. \quad (\text{A.21})$$

(2) For any $\vec{p}_n \in A_2 \times \dots \times A_g$ the matrix in (A.21) is not identically zero. **(3)** The two rows of the matrix in (A.21) are proportional to each other for any $\vec{p}_n \in A_2 \times \dots \times A_g$.

Lemma A.16. **(1)** The following identities hold for $j = 1, 2$:

$$N_j(\vec{p}_n) := -\frac{i}{\pi^2} \oint_{B_1} \operatorname{res}_{\varkappa=\tilde{\varkappa}_n} \Psi_{j1}(z; \varkappa) \operatorname{res}_{\varkappa=\tilde{\varkappa}_n} \Psi_{j2}(z; \varkappa) dz = \frac{\Theta(\mathbf{f}(\vec{p}_n) + (-1)^j 2\mathbf{u}(\infty))}{\Theta(W_0 + (-1)^j 2\mathbf{u}(\infty))} \frac{[\mathbb{A}^{-1} \nabla\Theta(W_0)]_g}{i\tilde{\tau}_1 \cdot \nabla\Theta(\mathbf{f}(\vec{p}_n))}. \quad (\text{A.22})$$

(2) The function $N_j(\vec{p})$ is a (real) analytic function of $\vec{p} \in A_2 \times \dots \times A_g$. It vanishes to second order at $p_{g-1} = \infty_l$, where ∞_l is the point at $z = \infty$ on the sheet $l = 1, 2$, and has no other zeroes.

According to (2.11), (2.2), the normalized singular function $\hat{f}_n(z)$ is propotional to

$$\varphi_{n,j}(z) = i\sqrt{w(z)} \operatorname{res}_{\lambda=\lambda_n} \Gamma_{j2}(z; \lambda) \frac{1}{\lambda}, \quad j = 1, 2, \quad (\text{A.23})$$

where at least one of the latter expressions is not zero. Note that $\varphi_{n,j}$ corresponds to the second term of $\phi_{n,j}$ from (2.11).

Proposition A.17. The norms in $L^2(I)$ of the singular functions $\phi_{n,j}$ are given by

$$\|\phi_{n,j}\|^2 = 2e^{(-1)^{j+1}2(d_\infty + \tilde{\varkappa}_n \mathbf{g}_\infty) - \tilde{\varkappa}_n} (\pi^2 N_j(\vec{p}_n) + \mathcal{O}(\tilde{\varkappa}_n^{-1})), \quad j = 1, 2. \quad (\text{A.24})$$

Moreover, $\|\varphi_{n,j}\|^2 = \frac{1}{2}\|\phi_{n,j}\|^2$, where $\|\varphi_{n,j}\|$ is the $L^2(I_i)$ norm of $\varphi_{n,j}$.

Corollary A.18. **(1)** The functions $N_j(\vec{p})$ have constant sign on the torus $A_2 \times \dots \times A_g$. The function $\sqrt{N_j(\vec{p})}$ can be defined analytically on the double cover of $A_2 \times \dots \times A_g$. **(2)** There exists $\nu > 0$ such that for all $\vec{p} \in A_2 \times \dots \times A_g$

$$\max_{j=1,2} |N_j(\vec{p})| > \nu. \quad (\text{A.25})$$

Corollary A.19. [Cor. 7.20, [BKT13]] Functions

$$\Upsilon_{j,k}(z; \vec{p}) = \sqrt{\frac{\Theta(W_0 + (-1)^j 2\mathbf{u}(\infty))}{\Theta(\mathbf{f}(\vec{p}) + (-1)^j 2\mathbf{u}(\infty))}} \times \frac{[\mathbb{A}^{-1} \nabla\Theta(W_0)]_g}{i\tilde{\tau}_1 \cdot \nabla\Theta(\mathbf{f}(\vec{p}))} \frac{\Theta((-1)^{k+1}\mathbf{u}(z) + (-1)^j \mathbf{u}(\infty) + \mathbf{f}(\vec{p}))r(z)}{\Theta((-1)^{k+1}\mathbf{u}(z) + (-1)^j \mathbf{u}(\infty) + W_0)}, \quad (\text{A.26})$$

$j, k = 1, 2$, are analytic in z on Z_0 and in \vec{p} on the double covering of the torus $A_2 \times \dots \times A_g$, where $Z_0 = \mathbb{C} \setminus [a_1, a_{2g+2}]$ together with the boundary points on both sides of the interval (a_1, a_{2g+2}) . Moreover, $\Upsilon_{1,k}(z; \vec{p})$ coincides with $\Upsilon_{2,k}(z; \vec{p})$, $k = 1, 2$, on $Z_0 \times A_2 \times \dots \times A_g$ modulo factor (-1) .

Remark A.20. Note that $(\Upsilon_{j,1})_+(z; \vec{p}_n) = \Upsilon^{(j)}(z; \mathbf{f}_n)$, the latter defined in (2.28), where $z \in I$. The subscript “+” indicates that the limiting value on the upper side of $z \in I$ in Z_0 is taken. In view of Corollary A.19, we denote by $\Upsilon_k(z; \vec{p})$ a function on $Z_0 \times A_2 \times \dots \times A_g$ that coincides (modulo sign) with both $\Upsilon_{1,k}(z; \vec{p})$ and $\Upsilon_{2,k}(z; \vec{p})$, $k = 1, 2$. Then for each $n \in \mathbb{N}$ we have $(\Upsilon_1)_+(z; \vec{p}_n) = \Upsilon(z; \mathbf{f}_n)$ on $z \in I$, see Theorem 2.10, modulo factor (-1) .

B Proofs of the technical lemmas

In this section, we use \widehat{f}_n to denote the n -th normalized singular function for the system (2.1), as well as its analytic continuation on $\mathbb{C} \setminus I_e$. It follows from (2.1) that each \widehat{f}_n is purely imaginary on I_i and defined uniquely modulo the factor -1 . According to (A.13),

$$\varphi_{n,j}(z) = i\sqrt{w(z)}e^{\tilde{\kappa}_n(\mathfrak{g}(z)-(-1)^j\mathfrak{g}_\infty)+(d(z)-(-1)^jd_\infty)} \left(\operatorname{res}_{\kappa=\tilde{\kappa}_n} \Psi_{j,2}(z; \kappa) + \mathcal{O}(\tilde{\kappa}_n^{-1}) \right) \quad (\text{B.1})$$

uniformly on any compact set not intersecting the lenses $\mathcal{L}_i^{(\pm)}$, $\mathcal{L}_e^{(\pm)}$, see Figure 3, and

$$\begin{aligned} \varphi_{n,j}(z) &= i\sqrt{w(z)}m_{n,j} \left[\operatorname{res}_{\kappa=\tilde{\kappa}_n} \left(\frac{\pm 1}{iw(z)} \Psi_{j1}(z) e^{-\kappa(\mathfrak{g}(z)+1)-d(z)} + \Psi_{j2}(z) e^{\kappa\mathfrak{g}(z)+d(z)} \right) + \mathcal{O}(\tilde{\kappa}_n^{-1}) \right], \\ \text{where } m_{n,j} &:= e^{-(-1)^j\tilde{\kappa}_n\mathfrak{g}(\infty)-(-1)^jd(\infty)}, \end{aligned} \quad (\text{B.2})$$

uniformly on any compact subset of $\mathcal{L}_i^{(\pm)}$ not containing the endpoints.

We now define the approximations $\varphi_{n,j}^{(\infty)}(z)$ of $\varphi_{n,j}(z)$ as $\varphi_{n,j}^{(\infty)} = i\sqrt{w(z)} \operatorname{res}_{\lambda=\lambda_n} \Gamma_{j2}^{(\infty)}(z; \lambda) \frac{1}{\lambda}$, $j = 1, 2$. Then, according to (A.16),

$$\varphi_{n,j}^{(\infty)}(z) = i\sqrt{w(z)}e^{\tilde{\kappa}_n(\mathfrak{g}(z)-(-1)^j\mathfrak{g}_\infty)+(d(z)-(-1)^jd_\infty)} \operatorname{res}_{\kappa=\tilde{\kappa}_n} \Psi_{j,2}(z; \kappa) \quad (\text{B.3})$$

for $z \in \mathbb{C} \setminus \bigcup_{\pm} \mathcal{L}_i^{(\pm)} \cup \mathcal{L}_e^{(\pm)}$ and

$$\varphi_{n,j}^{(\infty)}(z) = im_{n,j}\sqrt{w(z)} \operatorname{res}_{\kappa=\tilde{\kappa}_n} \left(\frac{\pm 1}{iw(z)} \Psi_{j1}(z) e^{-\kappa(\mathfrak{g}(z)+1)-d(z)} + \Psi_{j2}(z) e^{\kappa\mathfrak{g}(z)+d(z)} \right) \quad (\text{B.4})$$

for $z \in \mathcal{L}_i^{(\pm)}$ (we will not need an expression in $\mathcal{L}_e^{(\pm)}$).

Remark B.1. It follows from (2.24), (2.26) and that Schwarz symmetry of $\mathfrak{g}(z), d(z)$ that $\mathfrak{g}_+ + 1 = \frac{1}{2} + \frac{1}{2}(\mathfrak{g}_+ - \mathfrak{g}_-) = \frac{1}{2} + i\Im \mathfrak{g}_+$ on I_i and $d_+ = \frac{d_+ - d_- - \ln w}{2} = i\Im d_+ - \frac{1}{2} \ln w$ on I . Then, taking the + boundary value of (B.4) and using Proposition A.8, we obtain the following chain of equalities valid for $z \in I_i$ (we omit the dependence on z for brevity)

$$\begin{aligned} \varphi_{n,j}^{(\infty)}(z) &= im_{n,j}\sqrt{w} \operatorname{res}_{\kappa=\tilde{\kappa}_n} \left(\frac{1}{iw} \Psi_{j1+} e^{-\kappa(\mathfrak{g}_++1)-d_+} + \Psi_{j2+} e^{\kappa\mathfrak{g}_++d_+} \right) = \\ &= m_{n,j} \operatorname{res}_{\kappa=\tilde{\kappa}_n} \left(\Psi_{j1+} e^{-i\kappa\Im \mathfrak{g}_+ - \frac{\kappa}{2} - i\Im d_+} + i\overline{\Psi_{j2-}} e^{i\kappa\Im \mathfrak{g}_+ - \frac{\kappa}{2} + i\Im d_+} \right) = \\ &= m_{n,j} e^{-\frac{\tilde{\kappa}_n}{2}} \operatorname{res}_{\kappa=\tilde{\kappa}_n} \left(\Psi_{j1+} e^{-i\kappa\Im \mathfrak{g}_+ - i\Im d_+} - \overline{\Psi_{j1+}} e^{i\kappa\Im \mathfrak{g}_+ + i\Im d_+} \right) = \\ &= 2im_{n,j} e^{-\frac{\tilde{\kappa}_n}{2}} \Im \left(\operatorname{res}_{\kappa=\tilde{\kappa}_n} \Psi_{j1+} e^{-i\tilde{\kappa}_n\Im \mathfrak{g}_+ - i\Im d_+} \right). \end{aligned} \quad (\text{B.5})$$

Lemma B.2. *The functions $\prod_{j=1}^{2g+2} (z - a_j)^{\frac{1}{4}} \Upsilon_{jk}(z; \vec{p})$ are uniformly bounded on the compact $\bar{Z}_0 \times A_2 \times \dots \times A_g$.*

Proof. The closure \bar{Z}_0 of Z_0 include the endpoints a_j , $j = 1, \dots, 2g+2$. Lemma A.16 and Corollary A.19 imply

$$\Upsilon_{jk}(z; \vec{p}) = \frac{(-1)^j}{\sqrt{N_j(\vec{p})}} \times \frac{\Theta((-1)^{k+1}u(z) + (-1)^j u(\infty) + \mathbf{f}(\vec{p})) r(z)}{\Theta((-1)^{k+1}u(z) + (-1)^j u(\infty) + W_0)}. \quad (\text{B.6})$$

Moreover, according to Corollary A.19 and (A.25), we can always assume that for a given $\vec{p} \in A_2 \times \dots \times A_g$ we have $\left| \frac{1}{\sqrt{N_j(\vec{p})}} \right| < \nu^{-\frac{1}{2}}$. Thus, it remains to estimate the second factor in (B.6).

The numerator $\Theta((-1)^{k+1}u(z) + (-1)^j u(\infty) + \mathbf{f}(\vec{p}))$ is analytic (in all variables) on the compact set $(z; \vec{p}) \in \bar{Z}_0 \times A_2 \times \dots \times A_g$ and, thus, bounded there. The Theta function in the denominator depends only on z . According to Lemma A.6, it vanishes only at infinity (like z^{-1}) and at the $g-1$ points $z = a_j$, $j \in J$, where it vanishes like $\sqrt{z - a_j}$. Taking into account (2.27), we see that the ratio $\frac{r(z)}{\Theta((-1)^{k+1}u(z) + (-1)^j u(\infty) + W_0)}$ is bounded on \bar{Z}_0 away from the endpoints a_j , and behaves like $O(z - a_j)^{-\frac{1}{4}}$ near each endpoint a_j , $j = 1, \dots, 2g+2$. Thus, the statement of the lemma is proven. \square

Remark B.3. *According to Lemmas A.16, A.15,*

$$\Upsilon_{j,k}(z; \vec{p}_n) = \frac{\text{res}_{\varkappa = \varkappa_n} \Psi_{jk}(z; \varkappa)}{\pi \sqrt{N_j(\vec{p}_n)}}, \quad j, k = 1, 2. \quad (\text{B.7})$$

Thus, (B.5) implies that $\varphi_{n,j}^{(\infty)}$ belongs to $L^2(I_i)$.

Let \mathcal{J}^ω denote the ω neighborhood of the endpoints of I .

Lemma B.4. *For any $\omega > 0$ there exists some $c_\omega > 0$ such that*

$$|\hat{f}_n(z)| \leq \frac{c_\omega}{1 + |z|^{\frac{1}{2}}} e^{\varkappa_n(\Re g(z) + \frac{1}{2})} \quad \text{on } \bar{\mathbb{C}} \setminus \mathcal{J}^\omega. \quad (\text{B.8})$$

Proof. As it was mentined in Section A.1, $\hat{f}_n = \varphi_{n,j} / \|\varphi_{n,j}\|$, $j = 1, 2$, provided $\|\varphi_{n,j}\| > 0$. Note that for every $n \in \mathbb{N}$ at least one of $\|\varphi_{n,j}\| > 0$. Then, using the estimate (A.14) for $\mathcal{E}(z; \varkappa)$ and taking the residue of (A.13), we have

$$\hat{f}_n(z) = i e^{\varkappa_n(g(z) + \frac{1}{2}) + d(z)} \left(\sqrt{w(z)} \Upsilon_2(z; \vec{p}_n) + \frac{\sqrt{|w(z)|} \mathcal{O}(\varkappa_n^{-1})}{1 + |z|} \right) \quad (\text{B.9})$$

uniformly on closed subsets of $\mathbb{C} \setminus \mathcal{J}^\omega$. Here we also used Remarks B.3 and A.20. Near $z = \infty$ the function $\Upsilon_2(z; \vec{p}_n)$ has behavior

$$\Upsilon_2(z; \vec{p}_n) = \frac{K(\vec{p}_n)}{z} + \mathcal{O}(z^{-2}), \quad (\text{B.10})$$

see RHP A.5 and (B.7), with some constant $K(\vec{p}_n) > 0$. Since $K(\vec{p}_n)$ is continuous on the compact set $\vec{p}_n \in A_2 \times \dots \times A_g$, we conclude that there is $\hat{K} > 0$ and a neighborhood of $z = \infty$ such that

$|\Upsilon_2(z; \vec{p}_n)| \leq \frac{\widehat{K}}{|z|}$ in this neighborhood for all $n \in \mathbb{N}$. Then, according to Lemma B.2, there exists some $K > 0$, such that

$$|\Upsilon_2(z; \vec{p}_n)| \leq \frac{K}{1 + |z|} \quad (\text{B.11})$$

on $\mathbb{C} \setminus \mathcal{J}^\omega$ uniformly in $n \in \mathbb{N}$. The statement thus follows from (B.9). \square

Remark B.5. *The subtlety in proving the accuracy in (B.9) is that \widehat{f}_n is obtained by dividing (B.3) by (A.24) and, although $\max_{j=1,2} |N_j(\vec{p}_n)|$ is separated from zero, each of the sequences $|N_1(\vec{p}_n)|$, $|N_2(\vec{p}_n)|$ is, in general, not. However, for each $n \in \mathbb{N}$ we could always use the particular choice of j that provides the said maximum, which guarantees the uniformity of the estimate.*

The following corollary is a direct consequence of Corollary A.19.

Corollary B.6. *For any $\omega > 0$ and any closed interval $\mathcal{I} \subset \mathbb{R} \setminus \mathcal{J}^\omega$, the functions*

$$\mu_{\mathcal{I}}(\vec{p}) = \max_{z \in \mathcal{I}} |\Upsilon_2(z; \vec{p})|, \quad \nu_{\mathcal{I}}(\vec{p}) = \max_{z \in \mathcal{I}} \left| \frac{\partial}{\partial z} \Upsilon_2(z; \vec{p}) \right|, \quad (\text{B.12})$$

are continuous on $A_2 \times \cdots \times A_g$.

Proof of Lemma 3.5. Let us choose $\omega > 0$ so that $\mathbf{J} \setminus \mathcal{J}^\omega$ contains some segment \mathcal{I} . We construct intervals $J_n \subset \mathcal{I}$, $n \in \mathbb{N}$. In view of (B.9) and (2.12), it is sufficient to construct J_n so that $|\Upsilon_2(z; \mathbf{f}_n)|$ instead of $|f_n e^{-\varkappa_n(\mathfrak{g}(z) + \frac{1}{2})}|$ will be separated from zero on J_n (uniformly in n). Let

$$\mu_* = \min_{\vec{p} \in A_2 \times \cdots \times A_g} \mu_{\mathcal{I}}(\vec{p}), \quad \nu_* = \max_{\vec{p} \in A_2 \times \cdots \times A_g} \nu_{\mathcal{I}}(\vec{p}), \quad (\text{B.13})$$

and let the maximum $\mu_{\mathcal{I}}(\vec{p})$ of $|\Upsilon_2(z; \vec{p})|$ in $z \in \mathcal{I}$ be attained at some $z_{\vec{p}} \in \mathcal{I}$. Obviously, $\mu_* > 0$ and $\nu_* < \infty$. Let us now define the intervals J_n by

$$J_n = \left(z_{\vec{p}^{(n)}} - \frac{\mu_*}{2\nu_*}, z_{\vec{p}^{(n)}} + \frac{\mu_*}{2\nu_*} \right) \cap \mathcal{I}. \quad (\text{B.14})$$

Then the length of each J_n is at least $\min(\frac{\mu_*}{2\nu_*}, |\mathcal{I}|)$, where $|\mathcal{I}|$ is the length of \mathcal{I} . Then, according to (B.12), (B.13),

$$|\Upsilon_2(z; \vec{p}^{(n)})| \geq \frac{\mu_*}{2} \quad (\text{B.15})$$

for all $z \in J_n$. Thus, we completed the proof Lemma 3.5.

Proof of Lemma 4.2. The norm of $\varphi_{n,j}^{(\infty)}$ in $L^2(I_i)$ (here $\tilde{\lambda}_n = e^{-\tilde{\varkappa}_n}$ is the approximate singular-value) is given by

$$\int_{I_i} \left(\operatorname{res}_{\lambda=\tilde{\lambda}_n} \Gamma_{j2}^{(\infty)}(z; \lambda) \frac{1}{\lambda} \right)^2 w(z) dz = -i \int_{I_i} \left(\operatorname{res}_{\lambda=\tilde{\lambda}_n} \lambda J(\Gamma_{j1}^{(\infty)}(z; \lambda)) \frac{1}{\lambda} \right) \left(\operatorname{res}_{\lambda=\tilde{\lambda}_n} \Gamma_{j2}^{(\infty)}(z; \lambda) \frac{1}{\lambda} \right) dz, \quad (\text{B.16})$$

where $J(F) = F_+ - F_-$. We can thus deform the two contributions from the \pm boundary values to $\partial \mathcal{L}_i^{(\pm)}$ which consists of arcs joining the consecutive endpoints of I_i (in the formula below, we omit the reference to the dependence on \varkappa, z for brevity):

$$-i \int_{I_i} \left(\operatorname{res}_{\lambda=\tilde{\lambda}_n} \lambda J(\Gamma_{j1}^{(\infty)}(z; \lambda)) \frac{1}{\lambda} \right) \left(\operatorname{res}_{\lambda=\tilde{\lambda}_n} \Gamma_{j2}^{(\infty)}(z; \lambda) \frac{1}{\lambda} \right) dz =$$

$$\begin{aligned}
&= -im_{n,j}^2 e^{-\tilde{\kappa}_n} \sum_{\pm} (\pm) \int_{\partial \mathcal{L}_i^{(\pm)}} \operatorname{res}_{\kappa=\tilde{\kappa}_n} \Psi_{j1} e^{-\kappa \mathfrak{g}-d} \operatorname{res}_{\kappa=\tilde{\kappa}_n} \left(\frac{\pm 1}{iw} \Psi_{j1} e^{-\kappa(\mathfrak{g}+1)-d} + \Psi_{j2} e^{\kappa \mathfrak{g}+d} \right) dz = \\
&= -im_{n,j}^2 e^{-\tilde{\kappa}_n} \left(\oint_{B_1} \operatorname{res}_{\kappa=\tilde{\kappa}_n} \Psi_{j1}(z; \kappa) \operatorname{res}_{\kappa=\tilde{\kappa}_n} \Psi_{j2}(z; \kappa) dz + \right. \tag{B.17}
\end{aligned}$$

$$\left. + \sum_{\pm} \int_{\partial \mathcal{L}_i^{(\pm)}} \left(\operatorname{res}_{\kappa=\tilde{\kappa}_n} \Psi_{j1}(z; \kappa) \right)^2 \frac{e^{-\tilde{\kappa}_n(2\mathfrak{g}(z)+1)-2d(z)} dz}{iw(z)} \right). \tag{B.18}$$

The expression (B.17) is precisely $m_{n,j}^2 N_{n,j}$ from (A.22). The remaining terms on line (B.18) contribute to order $\mathcal{O}(\tilde{\kappa}_n^{-1})$ as we now explain. Indeed, according to (B.7) and Lemma B.2, there exists some $C > 0$, such that the integrals in (B.18) are bounded by

$$\int_{\partial \mathcal{L}_i^{(\pm)}} \frac{C}{|\sqrt{\prod_{j=1}^{2g+2} (z - a_j)}|} \frac{e^{-\tilde{\kappa}_n \Re(2\mathfrak{g}(z)+1)-2\Re d(z)}}{|w(z)|} |dz| \tag{B.19}$$

uniformly in $n \in \mathbb{N}$. Using $\Re(2\mathfrak{g}(z)+1) = C_j |z - a_j|^{\frac{1}{2}} (1 + \mathcal{O}(|z - a_j|))$, see (2.22), we can estimate (B.18) to be of order $\mathcal{O}(\kappa^{-1})$ as $\kappa \rightarrow +\infty$. Thus, according to Lemma A.16, we have proved

$$\left\| \varphi_{n,j}^{(\infty)} \right\|_{I_i}^2 = m_{n,j}^2 e^{-\tilde{\kappa}_n} \pi^2 N_j(\vec{p}_n) (1 + \mathcal{O}(\tilde{\kappa}_n^{-1})). \tag{B.20}$$

Using (4.1), (B.5) and (B.7), we obtain

$$\tilde{f}_n(z) = \frac{\varphi_{n,j}^{(\infty)}(z)}{m_{n,j} e^{-\frac{1}{2}\tilde{\kappa}_n} \pi \sqrt{N_j(\vec{p}_n)}}, \tag{B.21}$$

where the right hand side does not depend on $j = 1, 2$. Now (B.20), (B.21) and (1.12) imply $\|\tilde{f}_n(z)\|_{I_i} = 1 + \mathcal{O}(n^{-1})$.

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